

Solving a Cubic Polynomial

This is a general cubic polynomial equation:

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0$$

Instead of directly tackling the polynomial in this form, first consider a “smaller” form, then work backwards to show how it can be applied to the general form.

This smaller form is

$$x^3 = Ax + B.$$

It is missing the x^2 term. This is a *depressed* cubic. Any general cubic polynomial can be depressed by a horizontal shift of the input variable, which will be discussed later.

Assume the solution of the depressed form is

$$x = m + n.$$

Cube both sides and expand:

$$x^3 = (m + n)^3 = m^3 + 3m^2n + 3mn^2 + n^3.$$

Rearrange terms slightly:

$$x^3 = m^3 + n^3 + 3mn(m + n).$$

Since $x = m + n$, the equation is now

$$x^3 = m^3 + n^3 + 3mnx.$$

This is equivalent to the form $x^3 = Ax + B$, where $A = 3mn$ and $B = m^3 + n^3$.

Example: Solve $x^3 = x + 1$.

Solution: Set $3mn = 1$ and $m^3 + n^3 = 1$.

In the first equation, isolate n and substitute into the second equation:

$$n = \frac{1}{3m} \quad \rightarrow \quad m^3 + \left(\frac{1}{3m}\right)^3 = 1$$

Simplify the second equation by clearing fractions and collecting terms to one side:

$$\begin{aligned} m^3 + \frac{1}{27m^3} &= 1 \\ 27m^6 + 1 &= 27m^3 \\ 27m^6 - 27m^3 + 1 &= 0. \end{aligned}$$

Solve the last equation using the quadratic equation, where $m^6 = (m^3)^2$:

$$m^3 = \frac{-(-27) \pm \sqrt{(-27)^2 - 4(27)(1)}}{2(27)} = \frac{27 \pm \sqrt{621}}{54}.$$

Thus,

$$m = \left(\frac{27 \pm \sqrt{621}}{54} \right)^{1/3}.$$

Since $n = \frac{1}{3m}$, then

$$n = \frac{1}{3} \left(\frac{27 \pm \sqrt{621}}{54} \right)^{-1/3}.$$

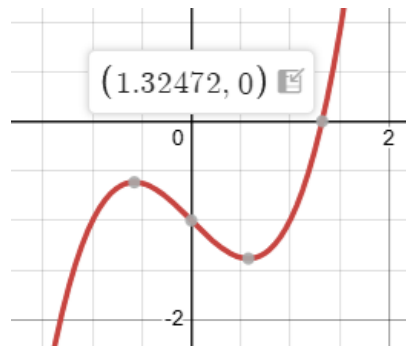
Lastly, since $x = m + n$, the solution to the equation $x^3 = x + 1$ is

$$x = \left(\frac{27 \pm \sqrt{621}}{54} \right)^{1/3} + \frac{1}{3} \left(\frac{27 \pm \sqrt{621}}{54} \right)^{-1/3}.$$

Using the positive radical gives:

$$x = \left(\frac{27 + \sqrt{621}}{54} \right)^{1/3} + \frac{1}{3} \left(\frac{27 + \sqrt{621}}{54} \right)^{-1/3}.$$

A calculator shows that this is $x = 1.324717957 \dots$



Graph of $f(x) = x^3 - x - 1$ with root highlighted.

Note that the negative radical form gives the same result:

$$x = \left(\frac{27 - \sqrt{621}}{54} \right)^{1/3} + \frac{1}{3} \left(\frac{27 - \sqrt{621}}{54} \right)^{-1/3} = 1.324717957 \dots$$

If C is a root of the equation $x^3 - Ax - B = 0$, then the cubic expression can be factored as

$$(x - C)(x^2 + Cx + C^2 - A).$$

The quadratic formula gives the two other roots:

$$x = \frac{-C \pm \sqrt{C^2 - 4(C^2 - A)}}{2} = \frac{-C \pm \sqrt{4A - 3C^2}}{2}.$$

In this example, the other two roots are $x = -0.6623589785 \pm 0.5622795116i$, both Complex.

In general, by setting $3mn = A$ and $m^3 + n^3 = B$, the general form can be found, as follows:

Solve for n in the first equation:

$$n = \frac{A}{3m}.$$

Substitute into the second equation, and simplify by clearing fractions and collecting terms to one side:

$$m^3 + \left(\frac{A}{3m}\right)^3 = B$$

$$m^3 + \frac{A^3}{27m^3} = B$$

$$27m^6 - 27Bm^3 + A^3 = 0$$

Using the quadratic formula gives

$$m^3 = \frac{-(-27B) \pm \sqrt{(-27B)^2 - 4(27)(A^3)}}{2(27)} = \frac{27B \pm \sqrt{729B^2 - 108A^3}}{54}.$$

Thus,

$$m = \left(\frac{27B \pm \sqrt{729B^2 - 108A^3}}{54}\right)^{1/3}.$$

Since $n = A/3m$, this gives

$$n = \frac{A}{3} \left(\frac{27B \pm \sqrt{729B^2 - 108A^3}}{54}\right)^{-1/3}$$

The general solution is

$$x = m + n = \left(\frac{27B \pm \sqrt{729B^2 - 108A^3}}{54}\right)^{1/3} + \frac{A}{3} \left(\frac{27B \pm \sqrt{729B^2 - 108A^3}}{54}\right)^{-1/3}.$$

As shown in the first example, either form with the positive or negative radicals gives the same result.

Simplified, the above form is

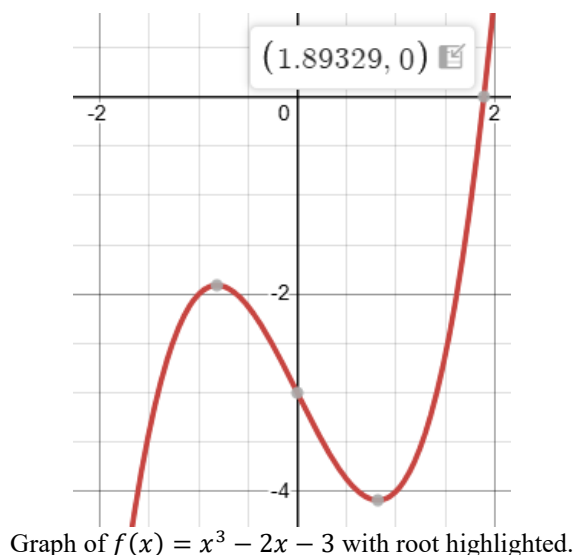
$$x = \left(\frac{1}{2}B \pm \frac{1}{18}\sqrt{81B^2 - 12A^3}\right)^{1/3} + \frac{A}{3} \left(\frac{1}{2}B \pm \frac{1}{18}\sqrt{81B^2 - 12A^3}\right)^{-1/3}.$$

Example: Solve $x^3 = 2x + 3$.

Solution: Note that $A = 2$ and $B = 3$. Thus,

$$\begin{aligned} x &= \left(\frac{1}{2}(3) + \frac{1}{18}\sqrt{81(3)^2 - 12(2)^3} \right)^{1/3} + \frac{(2)}{3} \left(\frac{1}{2}(3) + \frac{1}{18}\sqrt{81(3)^2 - 12(2)^3} \right)^{-1/3} \\ &= \left(\frac{3}{2} + \frac{1}{18}\sqrt{633} \right)^{1/3} + \frac{2}{3} \left(\frac{3}{2} + \frac{1}{18}\sqrt{633} \right)^{-1/3}. \end{aligned}$$

On a calculator, this is $x = 1.893289196 \dots$



Example: Solve $x^3 = -x + 2$.

Solution: Note that $A = -1$ and $B = 2$. By inspection, $x = 1$ is a solution. Will this method work? Substituting gives

$$\begin{aligned} x &= \left(\frac{1}{2}(2) + \frac{1}{18}\sqrt{81(2)^2 - 12(-1)^3} \right)^{1/3} + \frac{(-1)}{3} \left(\frac{1}{2}(2) + \frac{1}{18}\sqrt{81(2)^2 - 12(-1)^3} \right)^{-1/3} \\ &= \left(1 + \frac{1}{18}\sqrt{336} \right)^{1/3} - \frac{1}{3} \left(1 + \frac{1}{18}\sqrt{336} \right)^{-1/3}. \end{aligned}$$

On a calculator, this is $x = 1$. It worked. However, it is not obvious this long expression is 1 in disguise. This illustrates a “problem” with this method. It returns valid results, but in a form that can be hard to decipher. There is no easy way to simplify this expression.

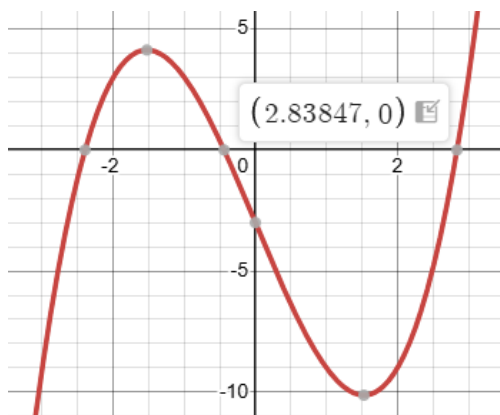
Example: Solve $x^3 = 7x + 3$.

Solution: Here, $A = 7$ and $B = 3$. Substituting gives

$$x = \left(\frac{1}{2}(3) + \frac{1}{18}\sqrt{81(3)^2 - 12(7)^3} \right)^{1/3} + \frac{(7)}{3} \left(\frac{1}{2}(3) + \frac{1}{18}\sqrt{81(3)^2 - 12(7)^3} \right)^{-1/3}$$

$$= \left(\frac{3}{2} + \frac{1}{18}\sqrt{-3387} \right)^{1/3} + \frac{7}{3} \left(\frac{3}{2} + \frac{1}{18}\sqrt{-3387} \right)^{-1/3}.$$

The radicand is negative, but interestingly, using a calculator gives $x = 2.838469252 + 0i$. The answer is Real, with no imaginary part.



Graph of $f(x) = x^3 - 7x - 3$ with root highlighted.

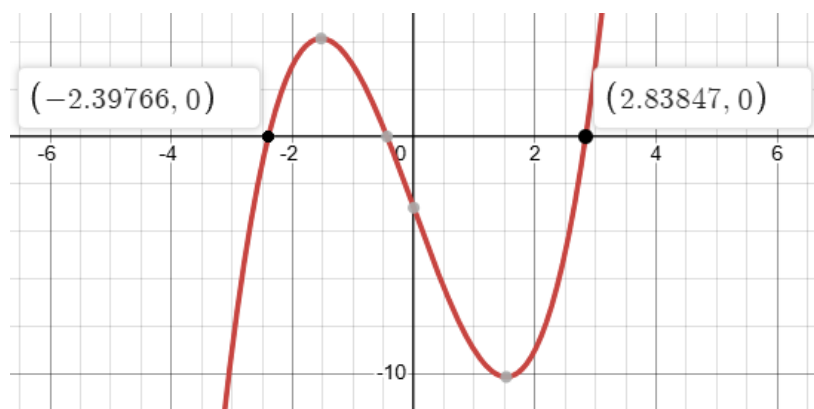
Observe that this method gives the largest possible solution. Suppose the graph is reflected across the y -axis. This is done by substituting $-x$ for x , and simplifying. The equation $x^3 = 7x + 3$, after substitution, becomes $(-x)^3 = 7(-x) + 3$, which is $-x^3 = -7x + 3$, or $x^3 = 7x - 3$. This is the same graph as the above but shifted up 6 units. Its roots will be negative of the three roots of the above graph.

Using the formula, only one small change happens: $B = -3$:

$$x = \left(\frac{1}{2}(-3) + \frac{1}{18}\sqrt{81(-3)^2 - 12(7)^3} \right)^{1/3} + \frac{(7)}{3} \left(\frac{1}{2}(-3) + \frac{1}{18}\sqrt{81(-3)^2 - 12(7)^3} \right)^{-1/3}$$

$$= \left(-\frac{3}{2} + \frac{1}{18}\sqrt{-3387} \right)^{1/3} + \frac{7}{3} \left(-\frac{3}{2} + \frac{1}{18}\sqrt{-3387} \right)^{-1/3}.$$

A calculator gives $x = 2.397661541 + 0i$. Thus, another root of the original equation is the negation, $x = -2.397661541$.



Graph of $f(x) = x^3 - 7x - 3$ with another root highlighted.

Referring to the formula for x ,

$$x = \left(\frac{1}{2}B \pm \frac{1}{18}\sqrt{81B^2 - 12A^3} \right)^{1/3} + \frac{A}{3} \left(\frac{1}{2}B \pm \frac{1}{18}\sqrt{81B^2 - 12A^3} \right)^{-1/3}.$$

The formula always gives a Real result, even in cases when the radicand $81B^2 - 12A^3$ is negative. When the radicand is positive, the graph of $f(x) = x^3 - Ax - B$ crosses the x -axis once, and has exactly one Real solution, the other two being Complex conjugate roots. When the radicand is zero or negative, the graph of f touches the x -axis twice (one root a repeated root) or three times (three Real roots).

Forcing the radicand to be positive gives

$$81B^2 - 12A^3 > 0.$$

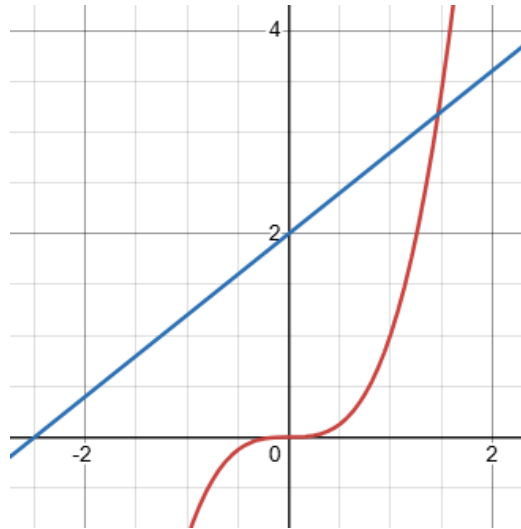
This is equivalent to

$$\frac{A^3}{B^2} < \frac{27}{4}.$$

Note that $\frac{81}{12} = \frac{27}{4}$ after simplification.

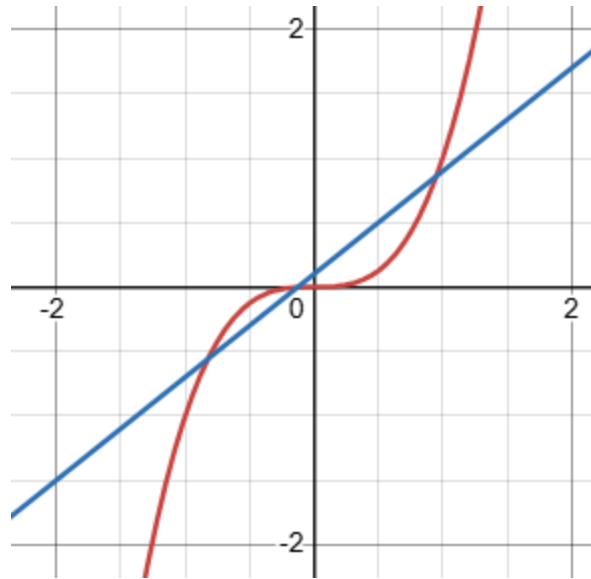
What does this actually mean? Is there a visual way to justify this? There is.

The graph below shows the basic cubic function $y = x^3$ with a generic linear graph $y = Ax + B$. Note that A is the slope of the line.



In this configuration, the graphs intersect once. Thus, the function $f(x) = x^3 - Ax - B$ will have one Real root and two Complex conjugate roots, and the radicand in the formula will be positive.

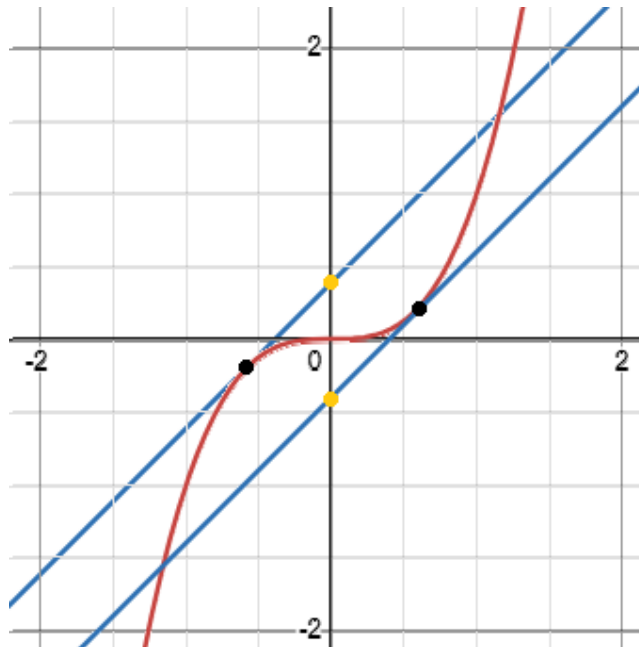
Suppose the linear graph moves downward, as shown below:



In this configuration, the graphs intersect three times. The function $f(x) = x^3 - Ax - B$ will have three Real roots, each of multiplicity 1 (that is, all unique). But more importantly, the radicand in the formula will be negative.

It is the B value that governs the line's y-axis intercept. Assume that A (the slope) is kept constant. Now find bounds for B that will ensure the radicand is positive and that the function $f(x) = x^3 - Ax - B$ has exactly one Real root. It is known that this happens when $\frac{A^3}{B^2} < \frac{27}{4}$.

Below, the graphs of the line $y = Ax + B$ touch tangentially to the graph of $y = x^3$ (the two black points). Where these lines cross the y-axis (the orange points) form an interval. If B is within this interval, then the radicand in the formula is negative and function $f(x) = x^3 - Ax - B$ has three unique Real roots. If B is outside this interval, then the radicand is positive and the function $f(x) = x^3 - Ax - B$ has exactly one Real root.



The derivative of $y = x^3$ is $y' = 3x^2$. Now find the x -values for which the slope of the tangent line is A ; that is, solve

$$3x^2 = A.$$

This gives

$$x = \pm \left(\frac{A}{3}\right)^{1/2}.$$

Consider the case for which $x = \left(\frac{A}{3}\right)^{1/2}$. The coordinate on the graph $y = x^3$ is $\left(\left(\frac{A}{3}\right)^{1/2}, \left(\frac{A}{3}\right)^{3/2}\right)$. The line has slope A , so using the point-slope formula for a line, $y - y_1 = m(x - x_1)$, and substituting gives

$$y - \left(\frac{A}{3}\right)^{3/2} = A\left(x - \left(\frac{A}{3}\right)^{1/2}\right).$$

Distribute and isolate y :

$$y = Ax - \frac{A^{3/2}}{3^{1/2}} + \left(\frac{A}{3}\right)^{3/2}.$$

The two constant terms can be simplified:

$$\begin{aligned} -\frac{A^{3/2}}{3^{1/2}} + \left(\frac{A}{3}\right)^{3/2} &= -\frac{A^{3/2}}{3^{1/2}} + \frac{A^{3/2}}{3^{3/2}} \\ &= -\frac{3 \cdot A^{3/2}}{3 \cdot 3^{1/2}} + \frac{A^{3/2}}{3^{3/2}} \\ &= -\frac{3A^{3/2}}{3^{3/2}} + \frac{A^{3/2}}{3^{3/2}} \\ &= \left(\frac{A}{3}\right)^{3/2} \cdot (-3 + 1) \\ &= -2\left(\frac{A}{3}\right)^{3/2}. \end{aligned}$$

Thus, the equation of the tangent line is

$$y = Ax - 2\left(\frac{A}{3}\right)^{3/2}.$$

The value $y = -2\left(\frac{A}{3}\right)^{3/2}$ is the y -intercept of the lower of the two lines. By symmetry, the value $y = 2\left(\frac{A}{3}\right)^{3/2}$ is the y -intercept of the upper of the two lines. This results in two statements:

- If $-2\left(\frac{A}{3}\right)^{3/2} < B < 2\left(\frac{A}{3}\right)^{3/2}$, then the graphs cross three times, the radicand in the formula is negative, and the function $f(x) = x^3 - Ax - B$ has three unique Real roots.
- If $B < -2\left(\frac{A}{3}\right)^{3/2}$ or $B > 2\left(\frac{A}{3}\right)^{3/2}$, then the graphs cross once, the radicand in the formula is positive, and the function $f(x) = x^3 - Ax - B$ has one Real root.

Consider the inequality $B > 2\left(\frac{A}{3}\right)^{3/2}$. It can be rearranged slightly. Divide by 2:

$$\frac{B}{2} > \left(\frac{A}{3}\right)^{3/2}$$

Square both sides:

$$\left(\frac{B}{2}\right)^2 > \left(\frac{A}{3}\right)^3.$$

Clear parentheses:

$$\frac{B^2}{4} > \frac{A^3}{27}.$$

Divide through by B^2 and A^3 , and rearrange:

$$\frac{A^3}{B^2} < \frac{27}{4}.$$

This was the same inequality derived from the radicand directly. Note that using any of the other expressions above will result in this same inequality. Lastly, note that if A is negative, then the above inequality is always true and there will only be a single Real root.

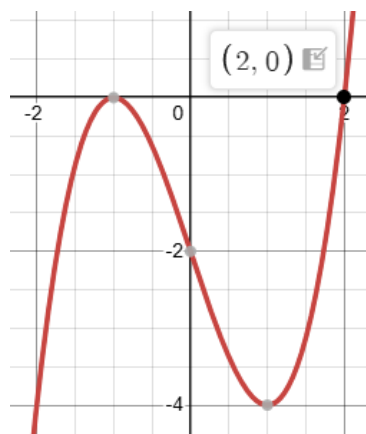
Consider the case when the radicand is 0:

Example: Solve $x^3 = 3x + 2$.

Solution: Note that $A = 3$ and $B = 2$. Thus,

$$\begin{aligned} x &= \left(\frac{1}{2}(2) + \frac{1}{18}\sqrt{81(2)^2 - 12(3)^3}\right)^{1/3} + \frac{(3)}{3}\left(\frac{1}{2}(2) + \frac{1}{18}\sqrt{81(2)^2 - 12(3)^3}\right)^{-1/3} \\ &= \left(1 + \frac{1}{18}\sqrt{0}\right)^{1/3} + \left(1 + \frac{1}{18}\sqrt{0}\right)^{-1/3}. \end{aligned}$$

This is just $1 + 1 = 2$, which is a solution that can be easily checked.



Graph of $f(x) = x^3 - 3x - 2$ with root $x = 2$ highlighted.

The expression $x^3 - 3x - 2$ factors as $(x - 2)(x + 1)^2$, so that $x = -1$ is the other root, with multiplicity 2.

The General Cubic Case

The general cubic equation is given by

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0.$$

The leading coefficient can always be divided out, resulting in a simpler form,

$$x^3 + bx^2 + cx + d = 0.$$

Out of laziness, $\frac{b}{a}$ is renamed b , $\frac{c}{a}$ is renamed c and $\frac{d}{a}$ is renamed d in the above equation. There is never any reason to not divide out the leading coefficient first.

The quadratic term can be removed by a horizontal shift. Let $x = w + h$, where h is the shift to be determined. Substituting,

$$(w + h)^3 + b(w + h)^2 + c(w + h) + d = 0.$$

Expanded, this is:

$$w^3 + 3w^2h + 3wh^2 + h^3 + bw^2 + 2bwh + bh^2 + cw + ch + d = 0.$$

The terms are recollected in powers of w :

$$w^3 + w^2(3h + b) + w(3h^2 + 2bh + c) + (h^3 + bh^2 + ch + d) = 0.$$

To remove the quadratic term (w^2), force $3h + b = 0$, which means that $h = -\frac{b}{3}$.

Thus, to depress a cubic equation of the form $x^3 + bx^2 + cx + d = 0$, set $x = w - \frac{b}{3}$.

Example: Find the Real root of $x^3 + 6x^2 - 7x + 10 = 0$.

Solution: Here, $b = 6$, so $h = -\frac{6}{3} = -2$.

Substituting this into the expanded form above, and noting that $c = -7$ and $d = 10$, results in:

$$w^3 + w^2(3(-2) + (6)) + w(3(-2)^2 + 2(6)(-2) + (-7)) + ((-2)^3 + (6)(-2)^2 + (-7)(-2) + (10)) = 0,$$

Which gives

$$w^3 + 0w^2 - 19w + 40 = 0.$$

This is now a depressed cubic:

$$w^3 - 19w + 40 = 0.$$

The graph of the depressed cubic $y = w^3 - 19w + 40$ is the graph of the original, $y = x^3 + 6x^2 - 7x + 10$, shifted to the right by 2 units.

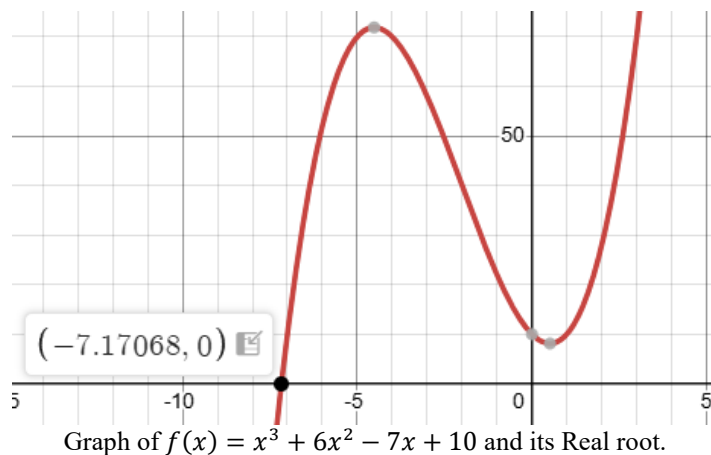
Now solve $w^3 - 19w + 40 = 0$ by writing it as $w^3 = 19w - 40$ and using the formula. Here, $A = 19$ and $B = -40$:

$$w = \left(\frac{1}{2}(-40) + \frac{1}{18}\sqrt{81(-40)^2 - 12(19)^3} \right)^{1/3} + \frac{(19)}{3} \left(\frac{1}{2}(-40) + \frac{1}{18}\sqrt{81(-40)^2 - 12(19)^3} \right)^{-1/3}$$

$$= \left(-20 + \frac{1}{18}\sqrt{425628} \right)^{1/3} + \frac{19}{3} \left(-20 + \frac{1}{18}\sqrt{425628} \right)^{-1/3}$$

This is $w = -5.170679557$.

The original substitution was $x = w - 2$, so that $x = -5.170679557 - 2 = -7.170679557$. This is the lone Real root of the original cubic equation.



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