Quiz 10 Key

Part A

Question 1 — 1 pts

Evaluate $\int_C F \cdot dr$ where $F(x,y) = \langle y^2, 3x \rangle$ and C is a sequence of line segments from (0,0) to (9,3) to (0,3) to (0,0).

Enter answer as a reduced fraction or decimal.

Path is a triangle traversed counter clockwise. Use Green's Theorem. Integrand is $N_x - M_y = 3 - 2y$. Thus, the double integral is $\int_0^9 \int_{x/3}^3 3 - 2y \; dy\, dx$. The inner integral is $\int_{x/3}^3 3 - 2y \; dy = [3y - y^2]_{x/3}^3 = (9-9) - \left(x - \frac{1}{9}x^2\right)$, which is $\frac{1}{9}x^2 - x$. The outer integral is $\int_0^9 \left(\frac{1}{9}x^2 - x\right) dx = \left[\frac{1}{27}x^3 - \frac{1}{2}x^2\right]_0^9 = \frac{729}{27} - \frac{81}{2} = -\frac{27}{2} = -13.5$.

Question 2 — 1 pts

Evaluate $\int_C F \cdot dr$ where $F(x,y) = \langle xy, 2x \rangle$ and C is the arc of a quarter circle of radius 5, centered at the origin, traversed counterclockwise from (5,0) to (0,5), then from there to the origin, then to (5,0).

Enter answer as a number to three decimal places

Use Green's Theorem. Integrand is $N_x - M_y = 2 - x$. Since the region of integration is a quarter circle, use polar. Thus, Green's Theorem gives $\int_0^{\pi/2} \int_0^5 (2 - r\cos\theta) r \; dr\, d\theta$. The inner integral is $\int_0^5 (2r - r^2\cos\theta) \; dr = \left[r^2 - \frac{1}{3}r^3\cos\theta\right]_0^5 = 25 - \frac{125}{3}\cos\theta$. The outer integral is $\int_0^{\pi/2} \left(25 - \frac{125}{3}\cos\theta\right) d\theta = \left[25\theta - \frac{125}{3}\sin\theta\right]_0^{\pi/2} = \frac{25}{2}\pi - \frac{125}{3}$, which is about -2.397.

Question 3 — 1 pts

Suppose there is a path in the xy-plane that starts at (1,1), then passes through (2,1), then to (5,2), then to (7,4), then back to (1,1).

Evaluate $\int_C F \cdot dr$ where $F(x,y) = \langle 2x + y, x + 2y \rangle$ and C is the part of the path that goes from (1,1) to (2,1) to (5,2) to (7,4).

Enter your answer as an integer.

The vector field is conservative. The potential function is $f(x,y) = x^2 + xy + y^2$. Evaluated at the end points gives $(7^2 + (7)(4) + 4^2) - (1^2 + (1)(1) + 1^2) = 49 + 28 + 16 - 3 = 90$.

Question 4 — 1 pts

Using your answer in the 3rd problem, what is the value of the line integral from (7,4) back to (1,1)?

Enter your answer as an integer.

The net work would be 0, so the remaining side would give a line integral value of -90.

Question 5

Which of the vector fields is conservative in R^3?

(A) $F(x,y,z) = \langle xy, 2x+z, y^2+3z \rangle$

(B) $F(x,y,z) = \langle y-3z, x+2z\ \ 2y-3x \rangle$

(C) $F(x,y,z) = \langle x-y^2, 2y+z, x-5z \rangle$

(D) $F(x,y,z) = \langle xy, xyz, yz \rangle$

A conservative vector field has curl F = 0. The answer is B.

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y-3z & x+2z & 2y-3z \end{vmatrix} = (2-2)i - (0-0)j + (1-1)k = \mathbf{0}.$$

Part B

In R^2, let vector field $F(x, y) = \langle 2x + y, 4x + 3y \rangle$ be given. Let C be a sequence of line segments from (0,0) to (6,0) to (6,5) to (0,5).

1. Note that C is not a closed loop. Suppose path D is the line segment from (0,5) to (0,0). Find the value of $\int_D F \cdot dr$ along this path.

2. Use Green's Theorem to find the value of the line integral around the hole loop, $\int_{C \cup D} F \cdot dr$.

3. Use your answers in (1) and (2) to determine the value of $\int_C F \cdot dr$.

1. Path D is parameterized as $r(t) = \langle 0, 5 - 5t \rangle, 0 \leq t \leq 1$. Thus $r'(t) = \langle 0, -5 \rangle$ and $F(t) = \langle 5 - 5t, 15 - 15t \rangle$. The integrand is $\langle 5 - 5t, 15 - 15t \rangle \cdot \langle 0, -5 \rangle = -75 + 75t$. The line integral is $\int_D F \cdot dr = \int_0^1 (-75 + 75t)\, dt$, which is $\left[-75t + \frac{75}{2}t^2\right]_0^1 = -75 + \frac{75}{2} = -\frac{75}{2}$ or -37.5.
2. The integrand is $N_x - M_y = 4 - 1 = 3$, so the integral is $\iint_R 3\, dA$, where R is the rectangle of base 6 and height 5, so the total line integral is $\int_{C \cup D} F \cdot dr = 3(area) = 3(6)(5) = 90$.
3. We have $\int_C F \cdot dr + \int_D F \cdot dr = \int_{C \cup D} F \cdot dr$. This is $\int_C F \cdot dr - 37.5 = 90$, so $\int_C F \cdot dr = 127.5$, or $\frac{255}{2}$.