Quiz 8 Key

Part A

Question 1 — 1 pts

A four-sided tetrahedral solid has vertices (0,0,0), (3,0,0), (0,6,0) and (0,0,12). A triple integral is to be set up using this solid as its region of integration. Which of the following triple integrals is correct?

(Select the letter of the correct set-up)

(A) $\int_0^3 \int_0^{6-3x} \int_0^{12-4x-2y} f(x,y,z)\, dz\, dy\, dx$

(B) $\int_0^{12} \int_0^{6-\frac{1}{2}z} \int_0^{3-\frac{1}{2}x-\frac{1}{4}y} f(x,y,z)\, dx\, dy\, dz$

(C) $\int_0^{12} \int_0^{12-4x} \int_0^{6-2x-\frac{1}{2}z} f(x,y,z)\, dy\, dz\, dx$

B is correct. The top bound in the dy integral in (A) is wrong, and the bounds in the outer (dx) integral in (C) is wrong.

Question 2 — 1 pts

A solid is bounded by the cylinder $x^2 + y^2 = 9$ within the first octant, and below the plane $y + z = 3$. Find its volume.

Enter your answer as a decimal approximation to three decimal places.

The integral is $\int_0^{\pi/2} \int_0^3 \int_0^{3-r\sin\theta} dz\, r\, dr\, d\theta$. The inner integral is $3 - r\sin\theta$, and the middle integral is $\int_0^3 3r - r^2 \sin\theta\ dr$ which evaluates to $\frac{3}{2}r^2 - \frac{1}{3}r^3 \sin\theta$ which then is $\frac{27}{2} - 9\sin\theta$ after the bounds are evaluated. The outer integral is $\int_0^{\pi/2} \left(\frac{27}{2} - 9\sin\theta\right) d\theta = \left[\frac{27}{2}\theta + 9\cos\theta\right]_0^{\pi/2} = \frac{27\pi}{4} + 0 - (0+9) = \frac{27\pi}{4} - 9 \approx 12.206.$

Question 3 — 1 pts

Consider this triple integral:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \left(x^2+y^2+z^2\right)\, dz\, dy\, dx$$

What is its equivalent integral in spherical form?

(Select the letter)

(A) $\int_0^{\pi} \int_0^{2\pi} \int_0^{1} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

(B) $\int_0^{2\pi} \int_0^{\pi} \int_0^{1} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

(C) $\int_0^{\pi} \int_0^{2\pi} \int_0^{1} \rho^4 \sin\phi \, d\rho \, d\theta \, d\phi$

(D) $\int_0^{\pi} \int_0^{\pi} \int_0^{1} \rho^4 \sin\phi \, d\rho \, d\theta \, d\phi$

The integrand is $x^2 + y^2 + z^2 = \rho^2$, which combined with the Jacobian gives $\rho^4 \sin\phi$. The object is a full sphere of radius 1, so the bounds are $0 \le \rho \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$. Thus. C is correct.

Question 4 — 1 pts

Evaluate the integral in the third problem.

Enter your answer as a decimal approximation to three decimal places.

The triple integral can be evaluated as three separate single integrals:

$$\int_0^{\pi} \sin\phi \, d\phi = [-\cos\phi]_0^{\pi} = (-\cos\pi) - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2,$$

$$\int_0^{2\pi} d\theta = 2\pi, \quad \int_0^{1} \rho^4 \, d\rho = \left[\frac{1}{5}\rho^5\right]_0^{1} = \frac{1}{5}.$$

Their product is $(2)(2\pi)\left(\frac{1}{5}\right) = \frac{4\pi}{5} \approx 2.513$

Part B

1. A cylinder $x^2 + y^2 = 16$ extends into the z direction, bounded by the planes $y + z = 4$ and $y - z = 4$. Use cylindrical coordinates/integration to find the volume of this object. Show your integral with proper bounds, and leave your answer in exact form. (3 pts)

The z bounds are $y - 4 \le z \le 4 - y$. The x and y variables are replaced with r and theta, with bounds $0 \le r \le 4, 0 \le \theta \le 2\pi$. The triple integral is $\int_0^{2\pi} \int_0^4 \int_{r\sin\theta - 4}^{4 - r\sin\theta} dz\, r\, dr\, d\theta$. The inner integral evaluates $8 - 2r\sin\theta$. The middle integral is $\int_0^4 8r - 2r^2 \sin\theta \; dr = 4r^2 - \frac{2}{3}r^3 \sin\theta$ which gives $64 - \frac{128}{3}\sin\theta$ after evaluation. The final integral is 2π so the object's volume is 128π. The cosine terms cancel after evaluation.

2. A sphere centered at the origin, and a cone with its vertex at the origin, intersect such that the point (2,1,5) lies on the edge where the two objects intersect. This is called a spherical sector. Find the object's volume using spherical coordinates and integration. Show the integral with proper bounds, and leave your answer in exact form (3 pts)

The object's radius is $r = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{30}$, so that the bounds on r are $0 \le r \le \sqrt{30}$. The object goes all the way around the z axis, so the theta bounds are $0 \le \theta \le 2\pi$. The phi bounds are $0 \le \phi \le \arccos\left(\frac{5}{\sqrt{30}}\right)$. The triple integral is

$$\int_0^{\arccos\left(\frac{5}{\sqrt{30}}\right)} \int_0^{2\pi} \int_0^{\sqrt{30}} \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi$$

These can be evaluated as three separate integrals:

$$\int_0^{\sqrt{30}} \rho^2 \, d\rho = \left[\frac{1}{3}\rho^3\right]_0^{\sqrt{30}} = \frac{1}{3}\left(\sqrt{30}\right)^3, \quad \int_0^{2\pi} d\theta = 2\pi,$$

$$\int_0^{\arccos\left(\frac{5}{\sqrt{30}}\right)} \sin\phi \; d\phi = [-\cos\phi]_0^{\arccos\left(\frac{5}{\sqrt{30}}\right)} = -\cos\left(\arccos\left(\frac{5}{\sqrt{30}}\right)\right) - (-\cos 0) = -\frac{5}{\sqrt{30}} + 1.$$

The object's volume is their product,

$$\frac{2\pi}{3}\sqrt{30}^3\left(1 - \frac{5}{\sqrt{30}}\right).$$

In this case, it's best to leave the phi bound in arccos form since it "plays nice" with the cosine after integration.

A slightly reduced form is

$$\frac{2\pi}{3}\left(\sqrt{30}^3 - 150\right).$$