Quiz 9 Key

Part A

Question 1

Which of the vector fields below is conservative?

(A) $F(x,y) = \langle 2x + y, x^2 - 2y \rangle$

(B) $F(x,y) = \langle 3x^2, 2xy \rangle$

(C) $F(x,y) = \langle \frac{x}{y}, -\frac{1}{y^2} \rangle$

(D) $F(x,y) = \langle 2xy^4 + 7, 4x^2y^3 - 8 \rangle$

Check $M_y = N_x$. The only one where this is true is (D).

Question 2 1

Evaluate $\int_C xy\,ds$, where C is the straight line from (2,1) to (7,13).

This is a **scalar** line integral! The path is $\mathbf{r}(t) = \langle 2 + 5t, 1 + 12t \rangle$ so that $\mathbf{r}'(t) = \langle 5,12 \rangle$ and $ds = |\mathbf{r}'(t)|dt = 13\,dt$. The bounds on t are 0 to 1. Since $x = 2 + 5t$ and $y = 1 + 12t$, then $xy = (2 + 5t)(1 + 12t) = 2 + 29t + 60t^2$. Thus, the integral is $\int_C\ 2x + y\,ds = 13\int_0^1 2 + 29t + 60t^2\,dt = 13\left[2t + \frac{29}{2}t^2 + 20t^3\right]_0^1 = 474.5, \text{or } \frac{949}{2}$.

Question 3 1 pts

Evaluate $\int_C F \cdot dr$, where $F(x,y) = \langle xy, x + y \rangle$ and C is the line segment from (1,2) to (5,4).

Enter your answer as a reduced fraction.

This is a **work** line integral! The path is $\mathbf{r}(t) = \langle 1 + 4t, 2 + 2t \rangle$ so that $\mathbf{r}'(t) = \langle 4,2 \rangle$. The vector field is $\mathbf{F}(x,y) = \langle xy, x + y \rangle = \langle (1 + 4t)(2 + 2t), (1 + 4t) + (2 + 2t) \rangle = \langle 2 + 10t + 8t^2, 3 + 6t \rangle$. The integrand is $\langle 2 + 10t + 8t^2, 3 + 6t \rangle \cdot \langle 4,2 \rangle = 8 + 40t + 32t^2 + 6 + 12t = 32t^2 + 52t + 14$. The integral is $\int_0^1 (32t^2 + 52t + 14)dt = \left[\frac{32}{3}t^3 + 26t^2 + 14t\right]_0^1 = \frac{32}{3} + 26 + 14 = \frac{152}{3}$.

Question 4

Evaluate $\int_C F \cdot dr$, where $F(x,y) = \langle x^2, y - x \rangle$ and C is the parabolic arc $y = x^2$ from (0,0) to (4,16).

Enter your answer as a reduced fraction.

This is a **work** line integral! The path is $\mathbf{r}(t) = \langle t, t^2 \rangle$ so that $\mathbf{r}'(t) = \langle 1, 2t \rangle$. Bounds are 0 to 4. Vector field is $\mathbf{F}(x,y) = \langle x^2, y - x \rangle = \langle t^2, t^2 - t \rangle$. The integrand is $\langle t^2, t^2 - t \rangle \cdot \langle 1, 2t \rangle = t^2 + 2t^3 - 2t^2 = 2t^3 - t^2$. The integral is $\int_0^4 (2t^3 - t^2)dt = \left[\frac{1}{2}t^4 - \frac{1}{3}t^3\right]_0^4 = 128 - \frac{64}{3} = \frac{320}{3}$.

Question 5 1 pt

Evaluate $\int_C F \cdot dr$, where $F(x,y) = \langle 6xy^3, 9x^2y^2 \rangle$ and C is the cubic arc $y = x^3$ from (0,0) to (2,8).

Enter your answer as an integer.

The vector field is conservative! The potential function is $f(x,y) = 3x^2y^3$. Just evaluate at the end points and subtract. At (2,8), you get $f(2,8) = 3(2)^2(8)^3 = 6{,}144$. At (0,0), you get 0. Thus, the line integral is 6,144 – 0 = 6,144.

Part B

1. Evaluate $\int_C 2x + y\,ds$ where C is the straight line segment from (2,-3) to (5,4).

2. Evaluate $\int_C F \cdot dr$ where $F(x,y) = \langle 3x, x - 2y\rangle$ and C is the parabolic arc $y = 2x^2$ from (0,0) to (2,8).

3. Evaluate $\int_C F \cdot dr$ where $F(x,y) = \langle -y, x + y\rangle$ and C is the half circle from (4,0) through (0,4) ending at (-4,0).

4. Evaluate $\int_C F \cdot dr$ where $F(x,y) = \langle 2xy + 3, x^2 + 2y\rangle$ and C is a parabolic arc from (0,0) to (3,9), then a line from (3,9) to (1,2).

5. Evaluate $\int_C F \cdot dr$ where $F(x,y) = \langle x^5 + 3y, 5x + y^7\rangle$ and C is a sequence of line segments from (0,0) to (6,0) to (3,4) back to (0,0)

1. The path is $\mathbf{r}(t) = \langle 2 + 3t, -3 + 7t\rangle$ so that $\mathbf{r}'(t) = \langle 3,7\rangle$ and $ds = |\mathbf{r}'(t)|dt = \sqrt{58}\,dt$. The bounds on t are 0 to 1. Since $x = 2 + 3t$ and $y = -3 + 7t$, then $2x + y = 2(2 + 3t) + (-3 + 7t) = 4 + 6t - 3 + 7t = 1 + 13t$. Thus, the integral is $\int_C 2x + y\,ds = \sqrt{58}\int_0^1 1 + 13t\,dt = \sqrt{58}\left[t + \frac{13}{2}t\right]_0^1 = \frac{15}{2}\sqrt{58}$.
2. The path is $\mathbf{r}(t) = \langle t, 2t^2\rangle$ so that $\mathbf{r}'(t) = \langle 1,4t\rangle$. The bounds on t are 0 to 2. Vector field $\mathbf{F}(x,y) = \langle 3x, x - 2y\rangle = \langle 3t, t - 4t^2\rangle$. Thus, $\mathbf{F} \cdot d\mathbf{r} = \langle 3t, t - 4t^2\rangle \cdot \langle 1,4t\rangle = 3t + 4t^2 - 16t^3$. The integral is $\int_0^2 (3t + 4t^2 - 16t^3)dt = \left[\frac{3}{2}t^2 + \frac{4}{3}t^3 - 4t^4\right]_0^2 = 6 + \frac{32}{3} - 64 = -\frac{142}{3}$.
3. The path is $\mathbf{r}(t) = \langle 4\cos t\,, 4\sin t\rangle$ so that $\mathbf{r}'(t) = \langle -4\sin t\,, 4\cos t\rangle$. The bounds on t are 0 to pi. Vector field $\mathbf{F}(x,y) = \langle -y, x + y\rangle = \langle -4\sin t\,, 4\cos t + 4\sin t\rangle$. The dot product is $\mathbf{F} \cdot d\mathbf{r} = \langle -4\sin t\,, 4\cos t + 4\sin t\rangle \cdot \langle -4\sin t\,, 4\cos t\rangle = 16\sin^2 t + 16\cos^2 t + 16\sin t\cos t = 16 + 16\sin t\cos t$. The integral is $\int_0^\pi 16 + 16\sin t\cos t\;dt = [16\theta + 8\sin^2 t]_0^\pi = 16\pi$. Note that the sine at pi and at 0 is 0 so all those terms vanish.
4. The vector field is conservative (Check My = Nx). The potential function is $f(x,y) = x^2y + 3x + y^2$. The endpoints are (1,2) and (0,0). Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = [x^2y + 3x + y^2]_{(0,0)}^{(1,2)} = 2 + 3 + 4 - 0 = 9$.
5. Use Green's Theorem. The integrand is $N_x - M_y = 5 - 3 = 2$, so that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\iint_R dA$. This is 2 times the area of the region the path encloses. It is a triangle of base 6 and height 4, so the line integral value is $2\iint_R dA = 2\left(\frac{1}{2}(6)(4)\right) = 24$.