The Chain Rule

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The Chain Rule is present in all differentiation.

If $z = f(x, y)$ represents a two-variable function, then it is plausible to consider the cases when x and y may be functions of other variable(s).

For example, consider the function $f(x, y) = x^2 + y^3$, where $x(t) = 2t + 1$ and $y(t) = 3t^2 + 4t$.

In such a case, we can find the derivative of f with respect to t by direct substitution, so that f is written as a function of t only, or we may use a form of the Chain Rule for multi-variable functions to find this derivative.

Example 1: Given $f(x, y) = x^2 + y^3$, where $x(t) = 2t + 1$ and $y(t) = 3t^2 + 4t$. Find $\frac{df}{dt}$.

Solution: Substitute $x(t) = 2t + 1$ and $y(t) = 3t^2 + 4t$ into the function f:

$$f(x(t), y(t)) = (2t + 1)^2 + (3t^2 + 4t)^3$$

Now, f is a function of t only. Expand by multiplication:

$$f(t) = 4t^2 + 4t + 1 + 27t^6 + 108t^5 + 144t^4 + 64t^3$$

Thus, $f(t) = 27t^6 + 108t^5 + 144t^4 + 64t^3 + 4t^2 + 4t + 1$. Its derivative is found by applying the Power Rule to each term:

$$\frac{df}{dt} = f'(t) = 162t^5 + 540t^4 + 576t^3 + 192t^2 + 8t + 4.$$

Now, let's try a different approach. Keeping the x and y variables present, write the derivative of f using the Chain Rule:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right) = (2x)(2) + (3y^2)(6t + 4).$$

Now we substitute $x(t) = 2t + 1$ and $y(t) = 3t^2 + 4t$ into the expression, and simplify:

$$\frac{df}{dt} = (2x)(2) + (3y^2)(6t + 4) = \big(2(2t + 1)\big)(2) + (3(3t^2 + 4t)^2)(6t + 4)$$

$$= 8t + 4 + 3(9t^4 + 24t^3 + 16t^2)(6t + 4) = 8t + 4 + 3(54t^5 + 180t^4 + 192t^3 + 64t^2)$$

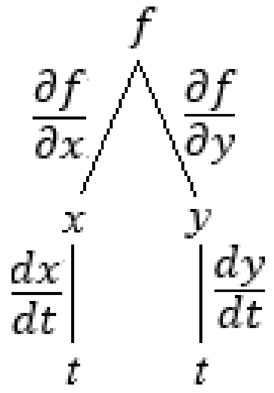
$$= 162t^5 + 540t^4 + 576t^3 + 192t^2 + 8t + 4.$$

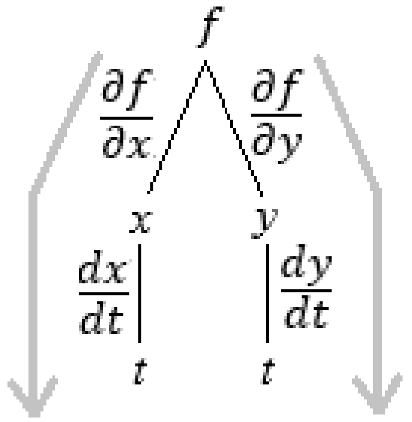
A useful way to visualize the form of the Chain Rule is to sketch a derivative tree.

In the previous example, we had f as a function of x and y, and then x and y as functions of t.

Thus, we would write the tree as shown below.

Then, the derivative form is found by multiplying along paths, and summing the separate paths:

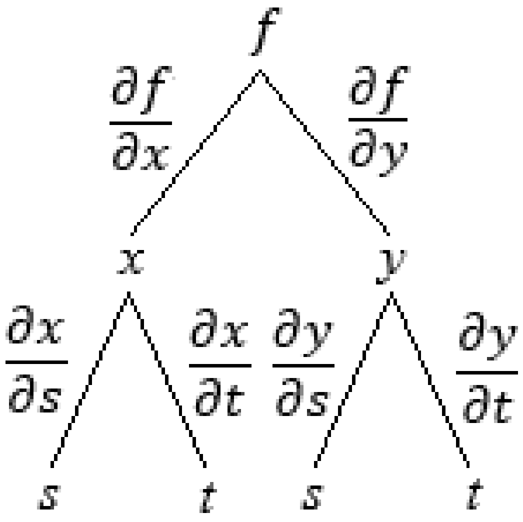




$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) \quad + \quad \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right)$$

Example 2: Suppose $f(x,y) = 2xy^2$ and $x(s,t) = 3s - 2t$ and $y(s,t) = s^2 + 4t$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

Solution: Note that f is a function of *x* and *y*, and that *x* and *y* are both functions of *s* and *t*. The derivative tree is shown below, with partial derivative notation attached to the "limbs":

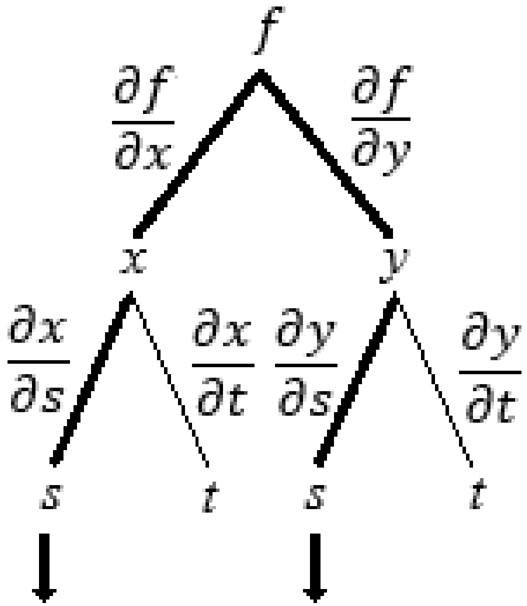


For example, the form of the partial derivative of f with respect to s is

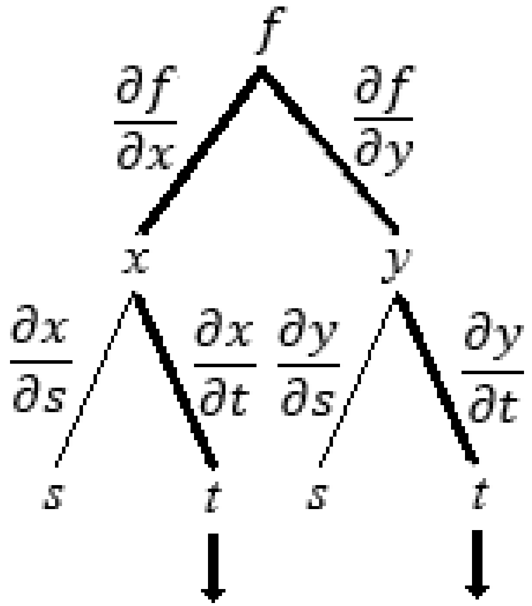
$$\frac{\partial f}{\partial s} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right).$$

In a similar way, the form of the partial derivative of f with respect to t is

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right).$$



$$\frac{\partial f}{\partial s} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right)$$



$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right)$$

To find $\frac{\partial f}{\partial s}$, we have

$$\frac{\partial f}{\partial s} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial s}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial s}\right)$$

$$= (2y^2)(3) + (4xy)(2s) = 6y^2 + 8xys$$

$$= 6(s^2 + 4t)^2 + 8(3s - 2t)(s^2 + 4t)s \quad \begin{cases} y = s^2 + 4t \\ x = 3s - 2t \end{cases}$$

This is simplified to

$$\frac{\partial f}{\partial s} = 30s^4 - 16s^3t + 144s^2t - 64st^2 + 96t^2.$$

To find $\frac{\partial f}{\partial t}$, we have

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial t}\right)$$

$$= (2y^2)(-2) + (4xy)(4) = -4y^2 + 16xy$$

$$= -4(s^2 + 4t)^2 + 16(3s - 2t)(s^2 + 4t) \qquad \begin{cases} y = s^2 + 4t \\ x = 3s - 2t \end{cases}$$

This simplifies to

$$\frac{\partial f}{\partial t} = -4s^4 + 48s^3 - 64s^2t + 192st - 192t^2.$$

Example 3: Let $y = g(u, v, w)$ and let u, v and w be functions of m and n. Suppose that $\frac{\partial g}{\partial m} = 19$, $\frac{\partial g}{\partial u} = 4$, $\frac{\partial g}{\partial v} = 2$, $\frac{\partial g}{\partial w} = 3$, $\frac{\partial u}{\partial m} = 5$, $\frac{\partial u}{\partial n} = 11$, $\frac{\partial v}{\partial n} = 1$, $\frac{\partial w}{\partial m} = -5$ and $\frac{\partial w}{\partial n} = 12$. Find $\frac{\partial v}{\partial m}$.

Solution: Using a derivative tree (or recognizing the pattern of the Chain Rule), we have

$$\frac{\partial g}{\partial m} = \left(\frac{\partial g}{\partial u}\right)\left(\frac{\partial u}{\partial m}\right) + \left(\frac{\partial g}{\partial v}\right)\left(\frac{\partial v}{\partial m}\right) + \left(\frac{\partial g}{\partial w}\right)\left(\frac{\partial w}{\partial m}\right).$$

By substitution, we have

$$19 = (4)(5) + (2)\left(\frac{\partial v}{\partial m}\right) + (3)(-5)$$

$$19 = 20 + 2\left(\frac{\partial v}{\partial m}\right) - 15$$

$$\frac{19 - 20 + 15}{2} = \frac{\partial v}{\partial m}$$

$$\frac{\partial v}{\partial m} = 7.$$

Implicit Differentiation can be performed by employing the chain rule of a multivariable function.

Often, this technique is much faster than the "traditional" direct method seen in single-variable calculus can be applied to functions of many variables with ease.

Example 4: Use implicit differentiation to find $\frac{dy}{dx}$ where $x^2y + y^3 = x^4$.

Solution: To use the Chain Rule, rewrite the equation $x^2y + y^3 = x^4$ with all terms to one side:

$$x^2y + y^3 - x^4 = 0.$$

Call the left side $F(x, y) = x^2y + y^3 - x^4$. We seek $\frac{dy}{dx}$, so differentiate both sides with respect to x. Using the Chain Rule, the derivative of F with respect to x is written $\left(\frac{\partial F}{\partial x}\right)\left(\frac{dx}{dx}\right) + \left(\frac{\partial F}{\partial y}\right)\left(\frac{dy}{dx}\right)$. Note that the right side gives us $\frac{d}{dx}0 = 0$. We have

$$\left(\frac{\partial F}{\partial x}\right)\left(\frac{dx}{dx}\right) + \left(\frac{\partial F}{\partial y}\right)\left(\frac{dy}{dx}\right) = 0.$$

Now, $\frac{\partial F}{\partial x} = 2xy - 4x^3$ and $\frac{\partial F}{\partial y} = x^2 + 3y^2$. Furthermore, $\frac{dx}{dx} = 1$, and $\frac{dy}{dx}$ is the unknown. Make the substitutions and solve for the unknown:

$$(2xy - 4x^3)(1) + (x^2 + 3y^2)\frac{dy}{dx} = 0$$

$$(x^2 + 3y^2)\frac{dy}{dx} = 4x^3 - 2xy$$

$$\frac{dy}{dx} = \frac{4x^3 - 2xy}{x^2 + 3y^2} .$$

In general, if x and y are implicitly related, collect all terms to one side and call the collected expression $F(x, y)$. Thus,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \qquad \text{and} \qquad \frac{dx}{dy} = -\frac{F_y}{F_x}\ .$$

This is true for implicit functions of three or more variables, too.

Example 5: Given $xy^2z + 3xz^3 = yz^6$, find $\frac{dy}{dz}$.

Solution: Call $F(x, y, z) = xy^2z + 3xz^3 - yz^6$. Using the formula $\frac{dy}{dz} = -\frac{F_z}{F_y}$, we have

$$F_z = xy^2 + 9xz^2 - 6yz^5 \qquad \text{and} \qquad F_y = 2xyz - z^6.$$

Thus,

$$\frac{dy}{dz} = -\frac{F_z}{F_y} = -\left(\frac{xy^2 + 9xz^2 - 6yz^5}{2xyz - z^6}\right) = \frac{6yz^5 - xy^2 - 9xz^2}{2xyz - z^6}\ .$$