Cross Product

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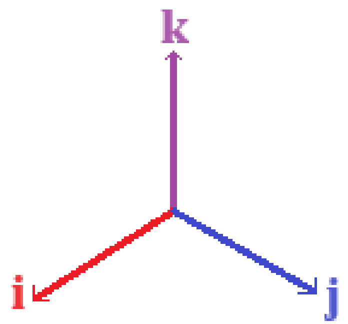
Start with the three orthonormal vectors $\mathbf{i} = \langle 1,0,0 \rangle$, $\mathbf{j} = \langle 0,1,0 \rangle$ and $\mathbf{k} = \langle 0,0,1 \rangle$.

By definition, we have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$.



By reflections, we have $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$.

Crossing a vector with itself results in the 0 vector: $\mathbf{i} \times \mathbf{i} = \mathbf{0}$, $\mathbf{j} \times \mathbf{j} = \mathbf{0}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$.

In R^3, let vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$.

Then, $\mathbf{u} \times \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$

This distributes out to:

$$u_1v_1(\mathbf{i} \times \mathbf{i}) + u_1v_2(\mathbf{i} \times \mathbf{j}) + u_1v_3(\mathbf{i} \times \mathbf{k}) + u_2v_1(\mathbf{j} \times \mathbf{i}) + u_2v_2(\mathbf{j} \times \mathbf{j})$$
$$+u_2v_3(\mathbf{j} \times \mathbf{k}) + u_3v_1(\mathbf{k} \times \mathbf{i}) + u_3v_2(\mathbf{k} \times \mathbf{j}) + u_3v_3(\mathbf{k} \times \mathbf{k}).$$

From the last screen we have

$$u_1v_1(\mathbf{i}\times\mathbf{i}) + u_1v_2(\mathbf{i}\times\mathbf{j}) + u_1v_3(\mathbf{i}\times\mathbf{k}) + u_2v_1(\mathbf{j}\times\mathbf{i}) + u_2v_2(\mathbf{j}\times\mathbf{j})$$
$$+u_2v_3(\mathbf{j}\times\mathbf{k}) + u_3v_1(\mathbf{k}\times\mathbf{i}) + u_3v_2(\mathbf{k}\times\mathbf{j}) + u_3v_3(\mathbf{k}\times\mathbf{k}).$$

Make replacements:

$$u_1v_1(\mathbf{0}) + u_1v_2(\mathbf{k}) + u_1v_3(-\mathbf{j}) + u_2v_1(-\mathbf{k}) + u_2v_2(\mathbf{0})$$
$$+u_2v_3(\mathbf{i}) + u_3v_1(\mathbf{j}) + u_3v_2(-\mathbf{i}) + u_3v_3(\mathbf{0}).$$

Collect:

$$(u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

The cross product is

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

The calculation of the cross product is best memorized as the determinant of a 3 by 3 matrix:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

Don't forget the negative in front of the **j** term.

The cross product is a vector that is simultaneously orthogonal to **u** and to **v**.

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0 \quad \text{and} \quad (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0.$$

Since it is easy to make a calculation error when finding the cross product, check your work by showing the result is orthogonal with the two original vectors.

Some of the common properties of the cross product are:

- Switching the order of the vectors results in a factor of –1: $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. The cross product is *not* commutative.

- Scalars can be grouped to the front: $c\mathbf{u} \times d\mathbf{v} = cd(\mathbf{u} \times \mathbf{v})$.

- Distributive property: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.

- The magnitude of the cross product is $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$. It is the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$. (Proof: https://www.surgent.net/math/mat267/cross_mag_proof.pdf)

- If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

- If vectors $\mathbf{u}$ and $\mathbf{v}$ are in R^2, then the vectors are rewritten as $\mathbf{u} = \langle u_1, u_2, 0\rangle$ and $\mathbf{v} = \langle v_1, v_2, 0\rangle$. In this case, the result is $\mathbf{u} \times \mathbf{v} = \langle 0, 0, u_1 v_2 - u_2 v_1\rangle$.

Example 1: Let $\mathbf{u} = \langle -1,4,6 \rangle$ and $\mathbf{v} = \langle 3,2,-4 \rangle$. Find $\mathbf{u} \times \mathbf{v}$.

Solution: We have

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & 6 \\ 3 & 2 & -4 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 2 & -4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 6 \\ 3 & -4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 4 \\ 3 & 2 \end{vmatrix} \mathbf{k}$$

$$= \big((4)(-4) - (6)(2)\big)\mathbf{i} - \big((-1)(-4) - (6)(3)\big)\mathbf{j} + \big((-1)(2) - (4)(3)\big)\mathbf{k}$$

$$= -28\mathbf{i} + 14\mathbf{j} - 14\mathbf{k}, \qquad \text{or} \quad \langle -28,14,-14 \rangle.$$

Check that this is correct by showing $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$ and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = 0$:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle -28, 14, -14 \rangle \cdot \langle -1, 4, 6 \rangle = (-28)(-1) + (14)(4) + (-14)(6)$$

$$= 28 + 56 - 84 = 0;$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle -28, 14, -14 \rangle \cdot \langle 3, 2, -4 \rangle = (-28)(3) + (14)(2) + (-14)(-4)$$

$$= -84 + 28 + 56 = 0.$$

Example 2: Let $\mathbf{u} = \langle -1,4,6 \rangle$ and $\mathbf{v} = \langle 3,2,-4 \rangle$. Find the area of the parallelogram formed by **u** and **v**, then find the area of the triangle formed by **u** and **v**.

Solution: From the previous example, we have $\mathbf{u} \times \mathbf{v} = \langle -28,14,-14 \rangle$. Thus, the area of the parallelogram formed by **u** and **v** is

$$|\mathbf{u} \times \mathbf{v}| = |\langle -28,14,-14 \rangle| = \sqrt{(-28)^2 + 14^2 + (-14)^2} = \sqrt{1176}, \text{ or about } 34.29 \text{ units}^2.$$

The area of the triangle formed by **u** and **v** is half this quantity, $\frac{1}{2}\sqrt{1176}$, or about 17.145 units^2.

Example 3: Find the area of the triangle formed by the points $A = (1,3,-2)$, $B = (4,0,3)$ and $C = (6,-3,5)$.

Solution: We form two vectors from among the three points (any pair of vectors will suffice). The vector from A to B is $\mathbf{AB} = \langle 3,-3,5\rangle$, and the vector from A to C is $\mathbf{AC} = \langle 5,-6,7\rangle$. The cross product of **AB** and **AC** is

$$\mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -3 & 5 \\ 5 & -6 & 7 \end{vmatrix} = \langle 9,4,-3\rangle.$$

(Be sure to verify that $(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AB} = 0$ and $(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AC} = 0$.)

The area of the triangle is half the magnitude of $\mathbf{AB} \times \mathbf{AC}$:

$$\text{Area} = \frac{1}{2}\sqrt{9^2 + 4^2 + (-3)^2} = \frac{1}{2}\sqrt{106} \approx 5.15 \text{ units}^2.$$

The Scalar Triple Product

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ be three vectors in R^3. The **scalar triple product** of **u**, **v** and **w** is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

The scalar triple product is a scalar quantity. Its absolute value is the volume of the **parallelepiped** (a “tilted” box) formed by **u**, **v** and **w**. The ordering of the vectors is not important. For example,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}), \qquad \mathbf{v} \cdot (\mathbf{u} \times \mathbf{w}), \qquad \text{and} \qquad \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}),$$

result in values that differ by at most the factor -1.

Example 4: Find the volume of the parallelepiped with sides represented by the vectors $\mathbf{u} = \langle -1,3,2 \rangle$, $\mathbf{v} = \langle 4,2,-5 \rangle$ and $\mathbf{w} = \langle 0,-3,1 \rangle$.

Solution: We have

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} -1 & 3 & 2 \\ 4 & 2 & -5 \\ 0 & -3 & 1 \end{vmatrix} = -1 \begin{vmatrix} 2 & -5 \\ -3 & 1 \end{vmatrix} - 3 \begin{vmatrix} 4 & -5 \\ 0 & 1 \end{vmatrix} + 2 \begin{vmatrix} 4 & 2 \\ 0 & -3 \end{vmatrix}$$

$$= -1(2 - 15) - 3(4 - 0) + 2(-12 - 0) = -23.$$

Taking the absolute value, the volume is 23 cubic units.

(The tetrahedron formed by **u**, **v** and **w** has a volume that is 1/6 of the volume of the parallelepiped.)

Work & Torque

Work is defined as force F (in Newtons) applied to move an object a distance of d (in meters). It is the product of F and d:

$$W = Fd$$

The standard metric unit for work is Joules, which is equivalent to Newton-meters.

If the force is not applied in the same direction that the object will move, then we need to find the component of **F** (written now as a vector) that is parallel to the direction **d**, also now written as a vector. Placing the feet of **F** and **d** together, the component of **F** in the direction of **d** is given by $|\mathbf{F}| \cos\theta$. If $|\mathbf{d}|$ is the length of vector **d** (not necessarily the distance moved by the object in the direction of **d**), then the work is given by

$$W = |\mathbf{F}||\mathbf{d}| \cos\theta \, .$$

This is the dot product of **F** and **d**. Thus, work can be defined as a dot product,

$$W = \mathbf{F} \cdot \mathbf{d}.$$

Work is a scalar value. It may be a negative value, which can be interpreted that the object is moving against the force being applied to it. For example, walking into a headwind at an angle.

Example 5: A force of 10 Newtons is applied in the direction of $\langle 1,1\rangle$ to an object that moves in the direction of the positive *x*-axis for 5 meters. Find the work performed on this object.

Solution: Using geometry, the component of the force in the direction of the positive *x*-axis is $|\mathbf{F}|\cos\left(\frac{\pi}{4}\right) = 10\left(\frac{\sqrt{2}}{2}\right) = 5\sqrt{2}$. The object moves 5 meters, so the work performed is

$$W = Fd = \left(5\sqrt{2}\right)(5) = 25\sqrt{2}\text{ Joules.}$$

Using vectors, the force vector is $\mathbf{F} = \left\langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}} \right\rangle$ and the direction vector is $\mathbf{d} = \langle 5,0\rangle$. Thus, the work performed is

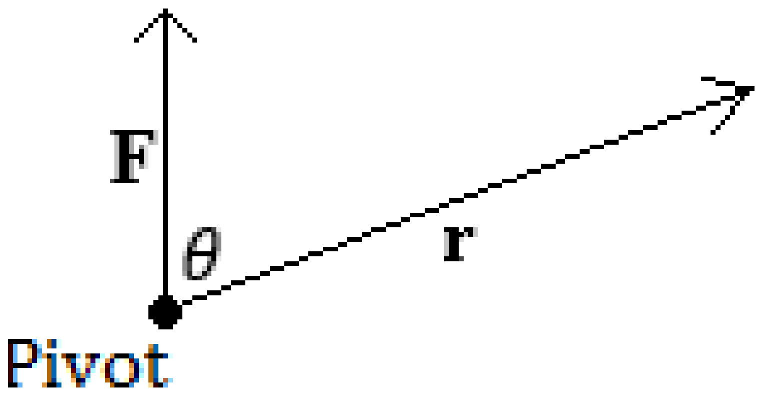
$$W = \mathbf{F}\cdot\mathbf{d} = \left\langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}} \right\rangle \cdot \langle 5,0\rangle = \frac{50}{\sqrt{2}} = 50\left(\frac{\sqrt{2}}{2}\right) = 25\sqrt{2}\text{ Joules.}$$

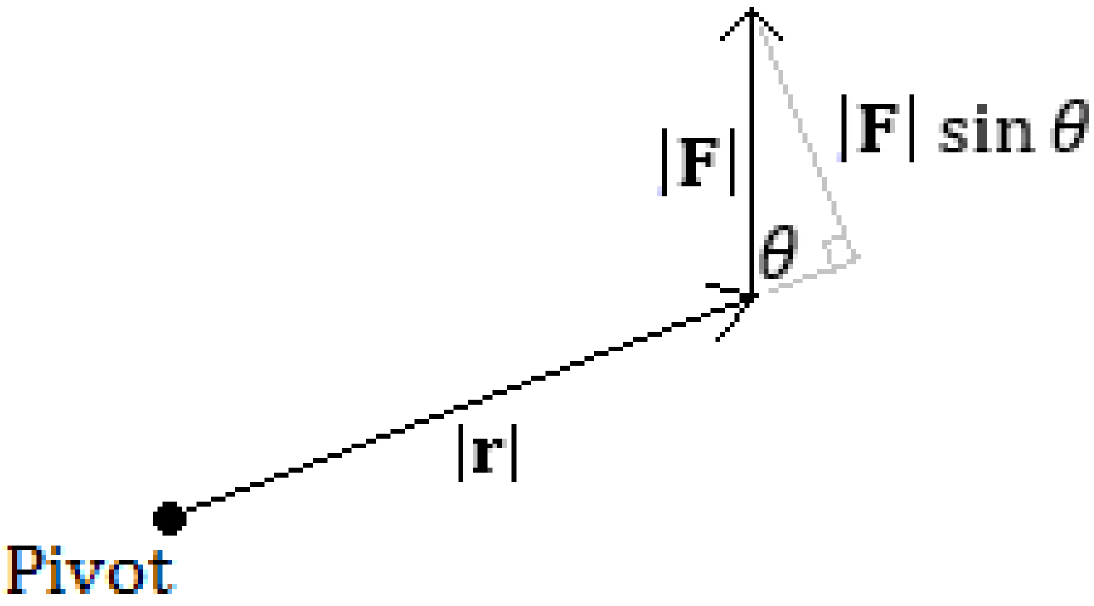
Torque describes the force resulting from a pivoting motion. For example, when a wrench is turned around a pivot point (*e.g.* a bolt), it creates a force in the direction of the bolt. If the wrench is described as a vector $\mathbf{r}$ (where the length of the wrench is $|\mathbf{r}|$) and the force applied to the wrench as another vector $\mathbf{F}$ (with magnitude $|\mathbf{F}|$), then the torque τ (tau) is orthogonal to both $\mathbf{r}$ and $\mathbf{F}$, defined by the cross product:

$$\tau = \mathbf{r} \times \mathbf{F}.$$

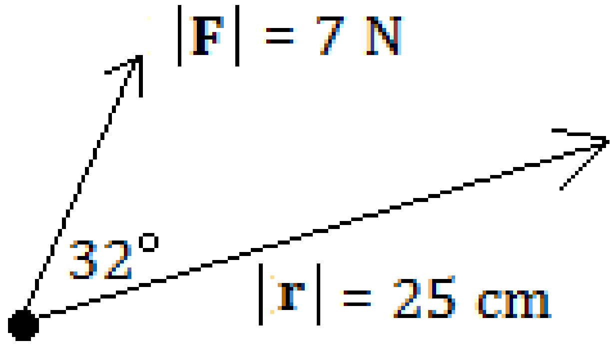
This means that torque τ is a vector. However, if both $\mathbf{r}$ and $\mathbf{F}$ lie in the *xy*-plane, then τ is a vector of the form $\langle 0,0,k \rangle$, and its magnitude is $|\tau| = k$. This scalar value is usually given as the torque in place of its vector form.

In the following images, vectors $\mathbf{r}$ and $\mathbf{F}$ are both drawn with their feet at a common point, the pivot point. The angle between $\mathbf{r}$ and $\mathbf{F}$ is θ. Naturally, we would not apply the force $\mathbf{F}$ at the pivot itself. Recall that a vector can be moved at will, so we locate $\mathbf{F}$ so that its foot is at $\mathbf{r}$'s head. Now we look for the component of the magnitude of $\mathbf{F}$ that is orthogonal to $\mathbf{r}$. We see that it is $|\mathbf{F}|\sin\theta$. If $|\mathbf{r}|$ is the length of the wrench, then the area given by $|\mathbf{F}||\mathbf{r}|\sin\theta$ is interpreted as the torque, as a scalar. This is $|\tau| = |\mathbf{r}\times\mathbf{F}|$.





Example 6: Find the torque around the pivot shown in the following diagram.



Solution: We convert the magnitude of **r** into meters. Thus, the torque is

$$|\tau| = |\mathbf{r} \times \mathbf{F}|$$

$$= |\mathbf{F}||\mathbf{r}| \sin \theta$$

$$= (7)(0.25) \sin 32^\circ \qquad (25 \text{ cm} = 0.25 \text{ m})$$

$$= 0.927 \text{ Nm}.$$