Triple Integration: Cylindrical & Spherical Coordinate Systems

Scott Surgent

This is a good one:

Example 1: Let solid S be a tetrahedron in the first octant with vertices (0,0,0), (2,0,0), (0,4,0) and (0,0,8). Set up a triple integral $\iiint_S f(x,y,z)\, dV$ and find the volume of the solid.

Solution: The equation of the plane that passes through the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ is given by

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad \text{(See Example 13.6 in my notes)}$$

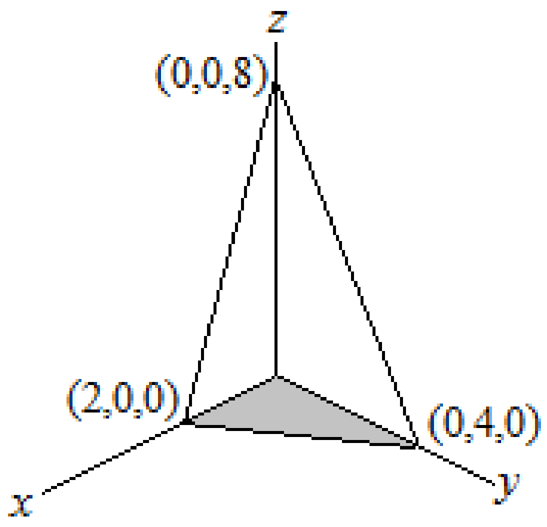
Thus, the equation of the plane passing through (2,0,0), (0,4,0) and (0,0,8) is

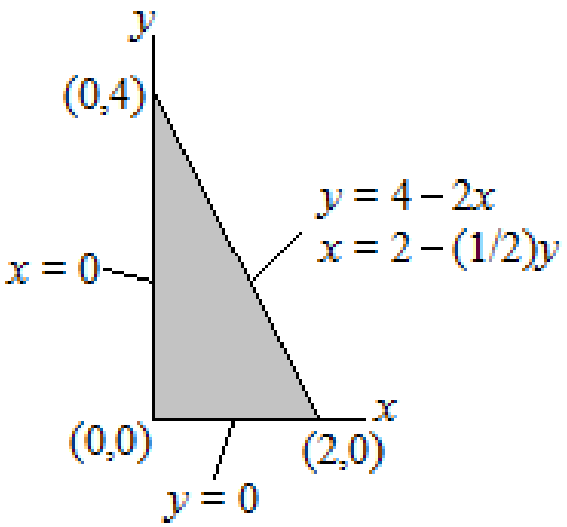
$$\frac{x}{2}+\frac{y}{4}+\frac{z}{8}=1.$$

If the inside integral is chosen to be evaluated with respect to *z*, then solve for *z*, getting $z = 8 - 4x - 2y$.

The bounds are $0 \le z \le 8 - 4x - 2y$.

This leaves a triangular region in the *xy*-plane with vertices (0,0), (2,0) and (0,4), shown below.





Integrating next with respect to *y*, the bounds are $0 \le y \le 4 - 2x$, and lastly, the bounds on x are $0 \le x \le 2$. The triple integral is

$$\int_0^2 \int_0^{4-2x} \int_0^{8-4x-2y} f(x,y,z)\, dz\, dy\, dx\,.$$

The volume of the tetrahedron is

$$\int_0^2 \int_0^{4-2x} \int_0^{8-4x-2y} 1 \, dz \, dy \, dx$$

The inner integral is

$$\int_0^{8-4x-2y} 1 \, dz = 8 - 4x - 2y.$$

This is then integrated with respect to y:

$$\int_0^{4-2x} (8 - 4x - 2y) \, dy = [8y - 4xy - y^2]_0^{4-2x}$$

$$= 8(4-2x) - 4x(4-2x) - (4-2x)^2 - 0$$

$$= 4x^2 - 16x + 16$$

This is then integrated with respect to x:

$$\int_0^2 (4x^2 - 16x + 16) \, dx = \left[\frac{4}{3}x^3 - 8x^2 + 16x\right]_0^2$$

$$= \frac{4}{3}(2)^3 - 8(2)^2 + 16(2) - 0$$

$$= \frac{32}{3} - 32 + 32$$

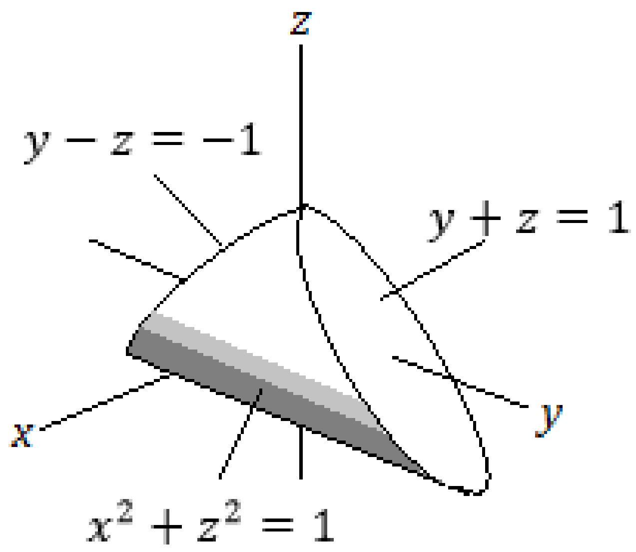
$$= \frac{32}{3}$$

Example 2: A cylinder, $x^2 + z^2 = 1$, is intersected by the planes $y + z = 1$ and $y - z = -1$. Find the volume of this intersecting region.

Solution: Below is a sketch of the region.

Note that the cylinder $x^2 + z^2 = 1$ can be viewed as a circle of radius 1, centered at the origin, on the *xz*-plane, then extended into the positive and negative *y* directions.

The planes $y + z = 1$ and $y - z = -1$ can be viewed as lines on the *yz*-plane, then extended into the positive and negative *x* directions.

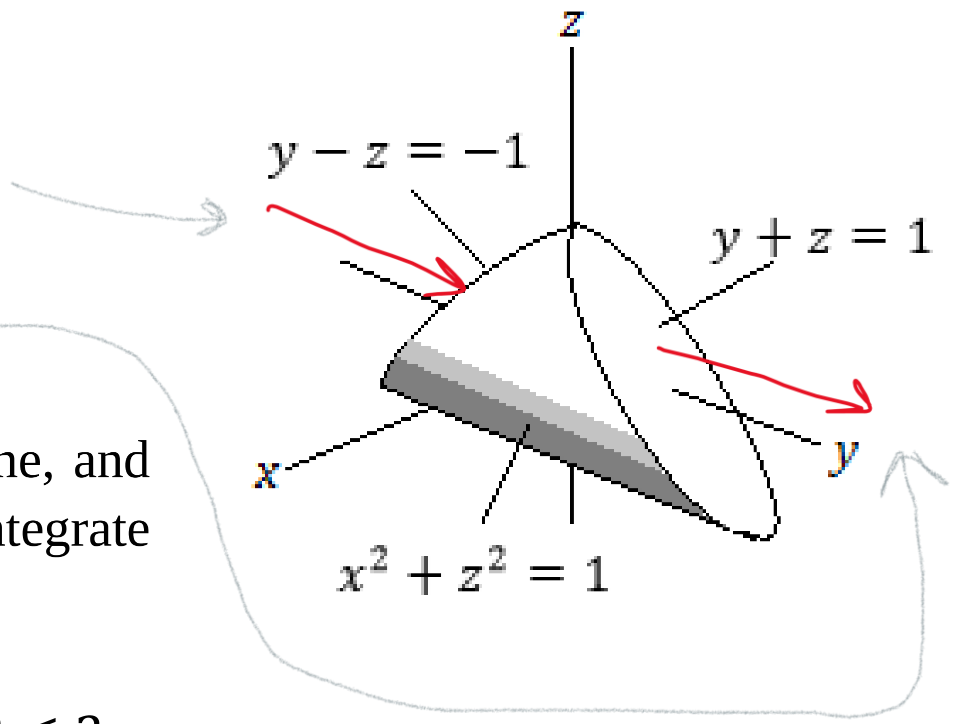


Visualize an arrow in the positive y direction.

It enters the solid through the plane $y - z = -1$, or $y_1 = z - 1$.

It exits the solid through the plane $y + z = 1$, or $y_2 = 1 - z$.

Note that variables x and z form a circular region on the xz-plane, and this suggests we may want to exchange them for r and θ, and integrate with respect to y first.



The bounds for r are $0 \leq r \leq 1$ and the bounds for θ are $0 \leq \theta \leq 2\pi$. An initial triple integral in cylindrical coordinates is given by

$$\int_0^{2\pi} \int_0^1 \int_{z-1}^{1-z} (1)\, dy\, r\, dr\, d\theta\,.$$

The bounds for y need to be written in terms of r and θ.

If we define $x = r\cos\theta$ and $z = r\sin\theta$, the triple integral is now written as

$$\int_0^{2\pi}\int_0^1\int_{r\sin\theta-1}^{1-r\sin\theta}(1)\,dy\,r\,dr\,d\theta\,.$$

The inside integral is evaluated first:

$$\int_{r\sin\theta-1}^{1-r\sin\theta}(1)\,dy = [y]_{r\sin\theta-1}^{1-r\sin\theta} = (1-r\sin\theta)-(r\sin\theta-1) = 2-2r\sin\theta\,.$$

This is integrated with respect to r:

$$\int_0^1 (2 - 2r\sin\theta)\, r\, dr = \int_0^1 (2r - 2r^2 \sin\theta)\ dr = \left[r^2 - \frac{2}{3}r^3 \sin\theta\right]_0^1 = 1 - \frac{2}{3}\sin\theta\,.$$

Finally, this is integrated with respect to θ:

$$\int_0^{2\pi} \left(1 - \frac{2}{3}\sin\theta\right)\ d\theta = \left[\theta + \frac{2}{3}\cos\theta\right]_0^{2\pi}$$

$$= \left((2\pi) + \frac{2}{3}\cos(2\pi)\right) - \left((0) + \frac{2}{3}\cos(0)\right) \qquad \begin{cases}\text{Recall that } \cos(2\pi) = 1 \\ \text{ and } \cos(0) = 1.\end{cases}$$

$$= 2\pi + \frac{2}{3} - \frac{2}{3} = 2\pi.$$

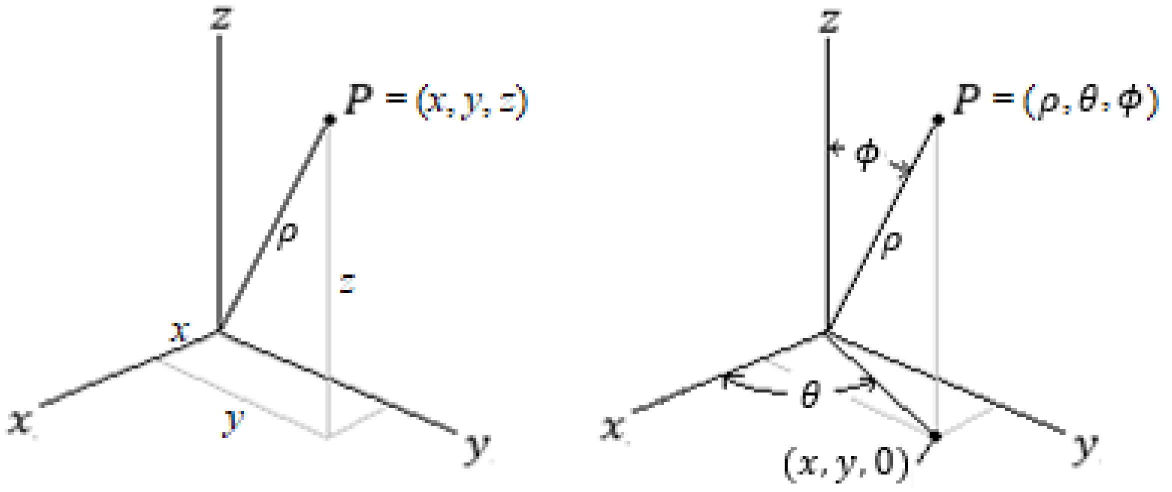
Spherical Coordinate System. A point $P = (x, y, z)$ described by rectangular coordinates in R^3 can also be described by three independent variables, ρ (rho), θ and ϕ (phi), whose meanings are given below:

ρ: the distance from the origin to P.

θ: the angle from the positive *x*-axis to the line connecting the origin to the point $(x, y, 0)$.

ϕ: the angle from the positive *z*-axis to the line connecting the origin to P.

Descriptively, ρ (rho) is the spherical radius, θ is the "sweep" or "azimuth" angle of the point's projection onto the *xy*-plane relative to the positive *x* axis, and ϕ (phi) is the "lean" angle of the point relative to the positive *z*-axis.



These three variables comprise the **spherical coordinate system** and are best used to describe regions in R^3 that are spheres, or parts of a sphere. For such regions, the bounds of ρ, θ and ϕ will be constants. The common restrictions on ρ, θ and ϕ are:

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

The variable ϕ can be thought of as the "lean" of the line connecting the origin to *P* relative to the positive *z*-axis.

If $\phi = 0$, then *P* lies on the positive *z*-axis.

If $\phi = \frac{\pi}{2}$, then *P* lies on the *xy*-plane, which is at right angles to the positive *z*-axis.

If $\phi = \pi$, then *P* lies on the negative *z*-axis.

The conversion formulas between rectangular coordinates (x, y, z) and spherical coordinates (ρ, θ, ϕ) are:

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad \phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right).$$

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

Example 3: Convert the rectangular coordinate $(2,5,3)$ into spherical coordinates.

Solution: This point lies above the first quadrant of the *xy*-plane. Thus, we expect that both θ and ϕ will be in the intervals $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$.

$$\rho = \sqrt{2^2 + 5^2 + 3^2} = \sqrt{38},$$

$$\theta = \arctan\left(\frac{5}{2}\right) \approx 1.1903 \text{ radians},$$

$$\phi = \arctan\left(\frac{\sqrt{2^2 + 5^2}}{3}\right) \approx 1.0625 \text{ radians}.$$

Since $\frac{\pi}{2} \approx 1.571$, the values for θ and ϕ are plausible.

Example 4: Convert the rectangular coordinate $(-3, -4, -2)$ into spherical coordinates.

Solution: This point lies below the third quadrant of the *xy*-plane. We expect that θ will be in the interval $\pi < \theta < \frac{3\pi}{2}$ and that ϕ will be in the interval $\frac{\pi}{2} < \phi < \pi$. We have

$$\rho = \sqrt{(-3)^2 + (-4)^2 + (-2)^2} = \sqrt{29}, \qquad \theta = \arctan\left(\frac{-4}{-3}\right) = \arctan\left(\frac{4}{3}\right) \approx 0.9273 \text{ radians},$$

$$\phi = \arctan\left(\frac{\sqrt{(-3)^2 + (-4)^2}}{-2}\right) = \arctan\left(-\frac{5}{2}\right) \approx -1.1903 \text{ radians}.$$

The current value for θ is incorrect. The value of 0.9273 radians places θ in the first quadrant. Thus, add π, so that the correct value for θ is 0.9273 + 3.1416 ≈ 4.0689 radians, which is in the in the interval $\pi < \theta < \frac{3\pi}{2}$, as desired.

Furthermore, we can rewrite ϕ so that it is in the interval $\frac{\pi}{2} < \phi < \pi$. We add π to $\phi \approx -1.1903$, getting –1.1903 + 3.1416 $\approx$ 1.9513 radians, which is an angle in the desired interval.

To summarize, the point $(-3, -4, -2)$ in rectangular coordinates is equivalent to the point $(\rho, \theta, \phi) = (\sqrt{29}, 4.0689, 1.9513)$ in spherical coordinates.

Example 5: Describe the solid sphere of radius 2 centered at the origin using spherical coordinates.

Solution: The solid sphere of radius 2 is described by

$$0 \leq \rho \leq 2, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Example 6: Describe $\rho = 3$, with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

Solution: This is a sphere of radius 3, centered at the origin. Had we set $0 \leq \rho \leq 3$, this would describe the solid sphere of radius 3.

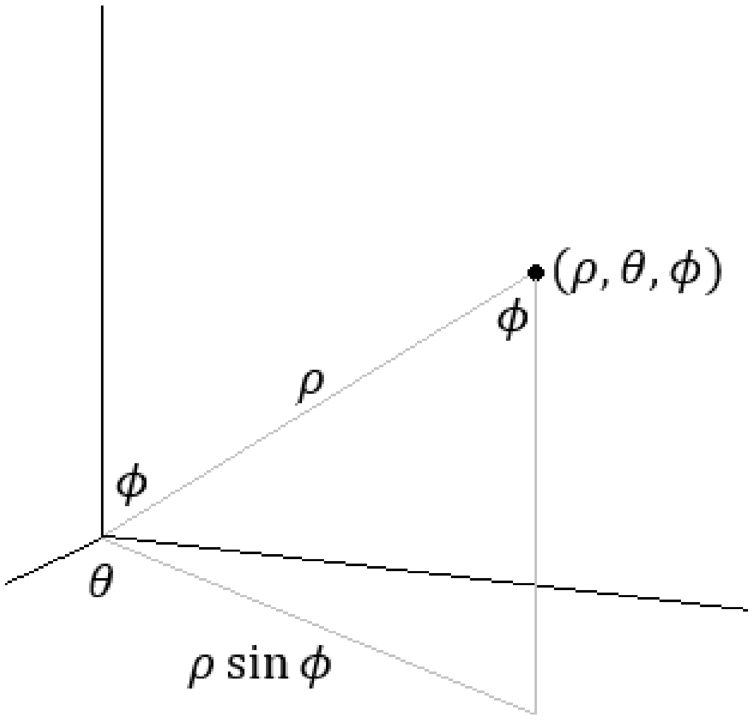
Converting back to rectangular coordinates, this same spherical surface is given by

$$x = 3 \sin \phi \cos \theta$$
$$y = 3 \sin \phi \sin \theta$$
$$z = 3 \cos \phi \,,$$

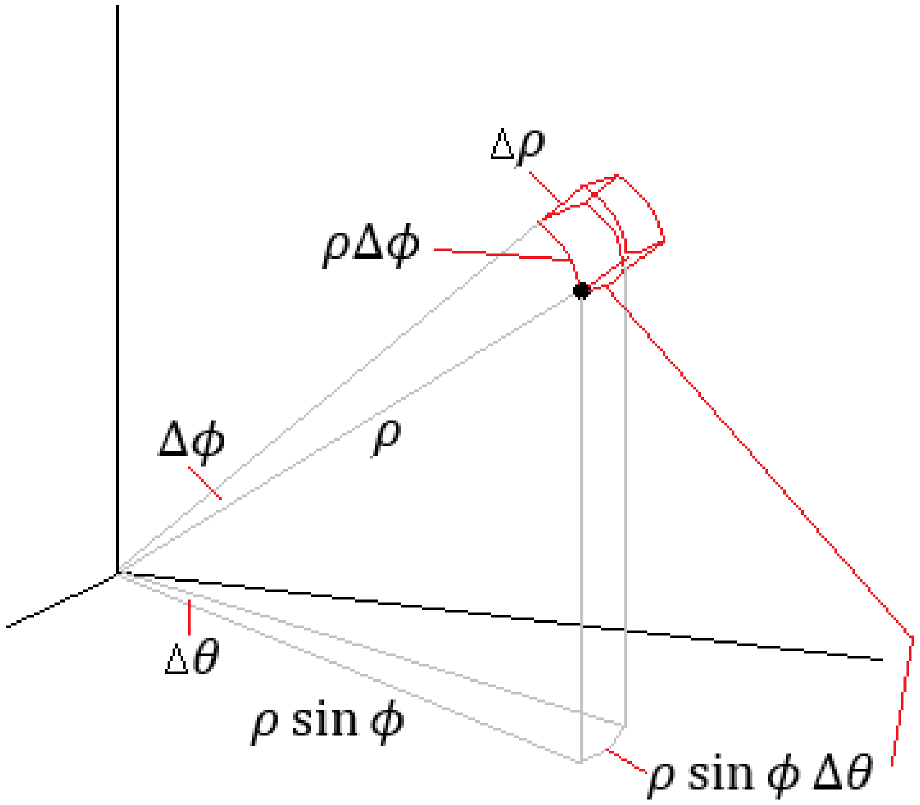
with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

The Jacobian of spherical integration.

Let (ρ, θ, ϕ) be a point in R^3.



Extend each value by a small amount, $\Delta\rho$, $\Delta\theta$ and $\Delta\phi$. This forms a “spherical rectangular solid”:



Use geometry to find the arc lengths as shown.

On small scales, the volume of this spherical volume element is the product of the three sides:

$$\Delta V = (\Delta\rho)(\rho \sin\phi \, \Delta\theta)(\rho\Delta\phi).$$

Using differentials, we have the Jacobian for Spherical Integrals:

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

“rho squared sine phi, d rho d theta d phi”

Example 7: Evaluate this integral:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx.$$

Solution: It is a sphere of radius 1.

From geometry, the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

Thus, we should expect the answer is $\frac{4}{3}\pi$.

In spherical coordinates, the bounds are $0 \le \rho \le 1,\ 0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.

The integral in spherical coordinates is

$$\int_{0}^{1} \int_{0}^{2\pi} \int_{0}^{\pi} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi.$$

From the last slide, we have

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

These are constant bounds and the integrand is held by multiplication. So we can treat this as the product of three single-variable integrals:

$$\left(\int_0^1 \rho^2 \, d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\right)$$

$$= \left(\left[\frac{1}{3}\rho^3\right]_0^1\right)\left([\theta]_0^{2\pi}\right)\left([-\cos\phi]_0^{\pi}\right)$$

$$= \left(\frac{1}{3}\right)(2\pi)\big(-(-1)-(-1)\big)$$

$$= \left(\frac{1}{3}\right)(2\pi)(2) = \frac{4}{3}\pi$$