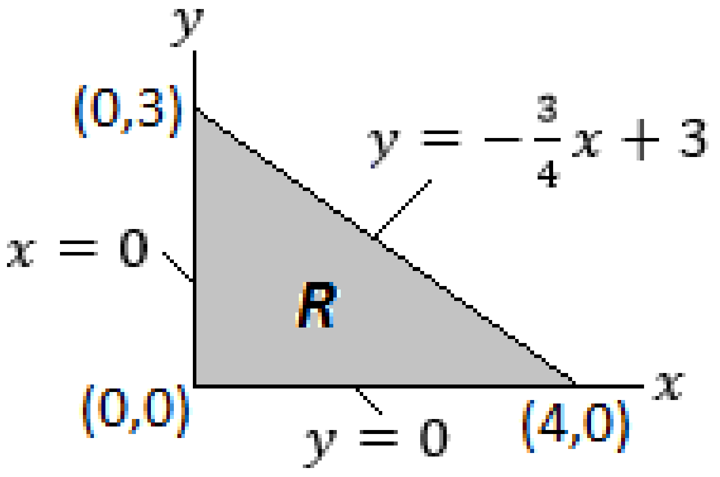
Integration over General Regions

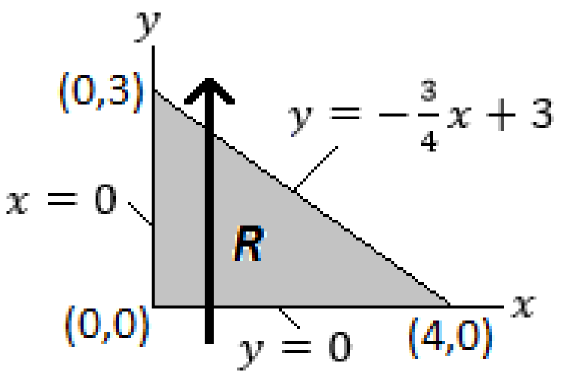
Scott Surgent

Consider the region R shown below.



The region is bounded by the lines $y = 0$ (the x-axis), $x = 0$ (the y-axis), and $y = -\frac{3}{4}x + 3$.

If we set up a double integral is the $dy\,dx$ ordering of integration, we draw an arrow in the positive y direction

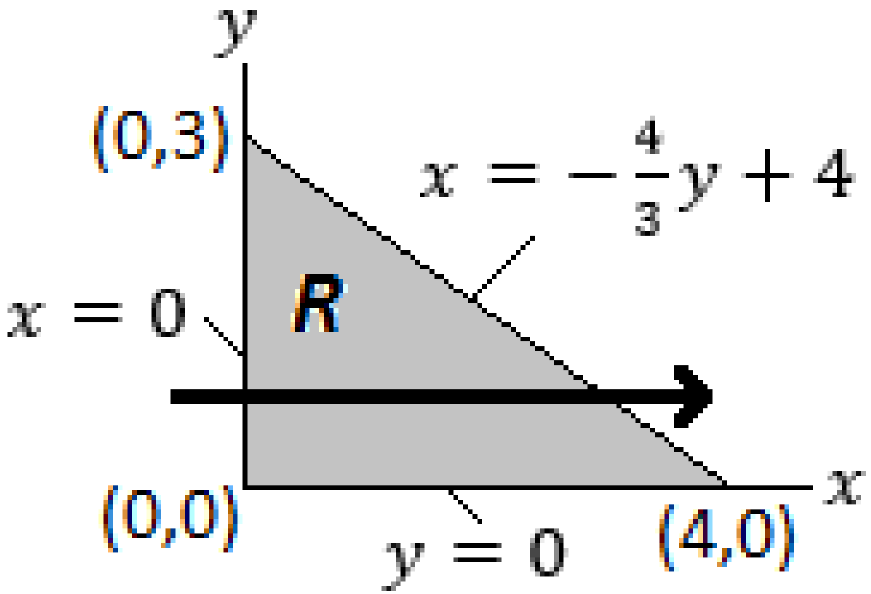


It enters the region at $y_1 = 0$ and exits through $y_2 = -\frac{3}{4}x + 3$, where the subscripts help us remember the order in which the boundaries are crossed.

The double integral is

$$\int_0^4 \int_{y_1=0}^{y_2=-(3/4)x+3} f(x,y)\ dy\ dx = \int_0^4 \int_0^{-(3/4)x+3} f(x,y)\ dy\ dx\,.$$

As a $dx\,dy$ integral, draw an arrow drawn in the positive x direction (see image at right).

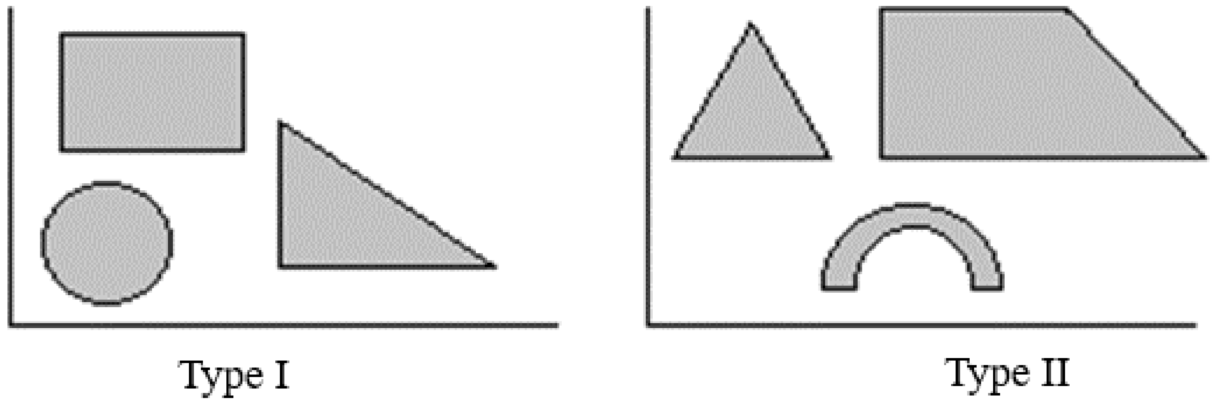


It enters the region at $x_1 = 0$ and exits through $x_2 = -\frac{4}{3}y + 4$ (which is the equation $y = -\frac{3}{4}x + 3$ that has been solved for x).

The resulting y bounds are 0 to 3, and the double integral is:

$$\int_0^3 \int_0^{-(4/3)y+4} f(x,y)\,dx\ \,dy\,.$$

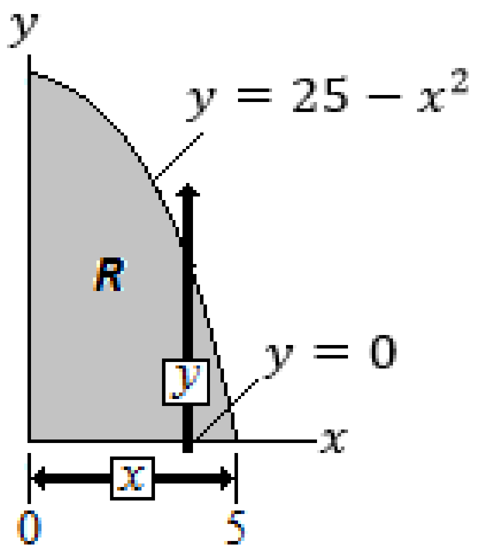
There is *no* ambiguity where an arrow enters or exits the region. Such a region is called a **Type I** region. If there is ambiguity, then the region is called a **Type II** region.



Example 1: Evaluate

$$\iint_R 2xy^2 \, dA,$$

where R is in the first quadrant bounded by the x-axis, the y-axis and the parabola $y = 25 - x^2$.



Solution: Sketch the region and decide on an ordering of integration.

If we choose a $dy\,dx$ ordering, visualize an arrow drawn in the positive *y* direction.

It enters the region at the *x*-axis, which is $y_1 = 0$, and exits through the parabola $y_2 = 25 - x^2$.

The bounds for *x* are 0 to 5.

The double integral is

$$\int_0^5 \int_0^{25-x^2} 2xy^2 \; dy\,dx.$$

The inside integral is determined:

$$\int_0^{25-x^2} 2xy^2 \ dy = \left[\frac{2}{3}xy^3\right]_0^{25-x^2} = \frac{2}{3}x(25-x^2)^3.$$

This is integrated with respect to *x* using a *u*-*du* substitution, with $u = 25 - x^2$:

$$\int_0^5 \frac{2}{3}x(25-x^2)^3 \ dx = \left[-\frac{1}{12}(25-x^2)^4\right]_0^5$$

$$= \left(-\frac{1}{12}(25-(5)^2)^4\right) - \left(-\frac{1}{12}(25-(0)^2)^4\right)$$

$$= 0 - \left(-\frac{1}{12}(25)^4\right) = \frac{390{,}625}{12} \ .$$

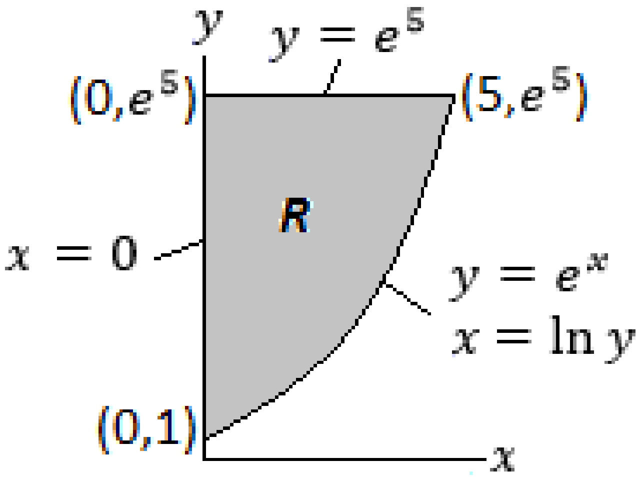
Example 2: Given

$$\int_0^5 \int_{e^x}^{e^5} g(x,y)\,dy\,dx,$$

Reverse the order of integration (that is, rewrite this double integral as a $dx\,dy$ integral).

Solution: The ordering of integration tells us that if we visualize an arrow in the positive y direction, it will enter the region at $y_1 = e^x$ and exit at the line $y = e^5$, with the x bounds being 0 to 5.

The region is shown with all vertices and boundaries identified.



To reverse the ordering, now visualize an arrow in the positive x direction.

It enters at $x_1 = 0$ (the y-axis) and exits at $x_2 = \ln y$. The bounds for y are 1 to e^5. We have

$$\int_1^{e^5} \int_0^{\ln y} g(x,y)\,dx\,dy.$$

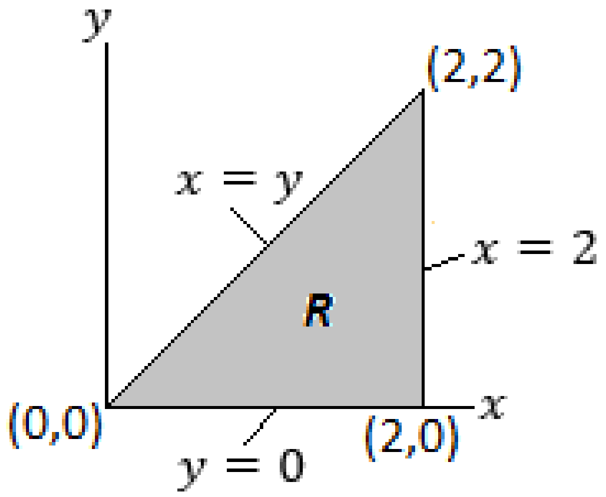
Example 3: Evaluate

$$\int_0^2 \int_y^2 \sqrt{1+x^2}\, dx\ dy\,.$$

Solution: If we attempt to evaluate the integrals as written (inside first with respect to x, then outside with respect to y), we discover that finding the antiderivative of $\sqrt{1+x^2}$ with respect to x is challenging (it would require a trigonometric substitution).

Instead, we reverse the order of integration.

The double integral, as written, suggests that the region R is bounded by the line $x = y$ and the line $x = 2$, with the bounds for y being 0 to 2. This region is sketched below, and all vertices and boundaries are identified:



Reversing the order of integration, we visualize an arrow in the positive y-direction. It enters R at $y_1 = 0$ and exits at $y_2 = x$. The bounds for x will be 0 to 2, and the double integral in the $dy\,dx$ ordering is

$$\int_0^2 \int_0^x \sqrt{1+x^2}\; dy\; dx.$$

Now, the inside integral is determined. Note that the antiderivative of $\sqrt{1+x^2}$ with respect to y is $y\sqrt{1+x^2}$. Thus, we have

$$\int_0^x \sqrt{1+x^2}\; dy = \left[y\sqrt{1+x^2}\right]_0^x = x\sqrt{1+x^2}.$$

Now we integrate $x\sqrt{1+x^2}$ with respect to x. The antiderivative of $x\sqrt{1+x^2}$ is found by a *u*-*du* substitution. We have

$$\int_0^2 x\sqrt{1+x^2}\; dx = \left[\frac{1}{3}(1+x^2)^{3/2}\right]_0^2 = \frac{1}{3}\left(5^{3/2}-1\right).$$