Equations of Lines & Planes

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Lines in R^3

Given a point $P_0 = (x_0, y_0, z_0)$ and a direction vector $\mathbf{v}_1 = \langle a, b, c \rangle$ in R^3, a line L that passes through P_0 and is parallel to $\mathbf{v}$ is written parametrically as a function of t:

$$x(t) = x_0 + at, \qquad y(t) = y_0 + bt, \qquad z(t) = z_0 + ct$$

Using vector notation, the same line is written

$$\begin{aligned} \langle x, y, z \rangle &= \mathbf{v}_0 + t\mathbf{v}_1 \\ &= \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle, \end{aligned}$$

where $\mathbf{v}_0$ is the vector whose head is located at $P_0 = (x_0, y_0, z_0)$.

Example 1: Find the parametric equation of a line passing through $P_0 = (2, -1, 3)$ and parallel to the vector $\mathbf{v}_1 = \langle 5,8,-4 \rangle$.

Solution: The line is represented parametrically by

$$x(t) = 2 + 5t, \qquad y(t) = -1 + 8t, \qquad z(t) = 3 - 4t,$$

or in vector notation as

$$\langle x, y, z \rangle = \langle 2, -1,3 \rangle + t\langle 5,8,-4 \rangle = \langle 2 + 5t, -1 + 8t, 3 - 4t \rangle.$$

Note that when $t = 0$, we obtain the vector $\langle 2, -1,3 \rangle$. If the foot of this vector is placed at the origin, then its head is the ordered triple $P_0 = (2, -1, 3)$

A **line segment** from a point $P_0 = (x_0, y_0, z_0)$ to a point $P_1 = (x_1, y_1, z_1)$ over $a \leq t \leq b$ has the form

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \frac{t - a}{b - a} \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle,$$

Note that $\langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ is the direction vector of the line.

Example 2: Find the parametric equation of the line segment from $P_0 = (4, 2, -1)$ to $P_1 = (7, -3, -2)$.

Solution: The direction vector $\mathbf{v}_1$ is found by subtracting P_0 from P_1:

$$\mathbf{v}_1 = P_1 - P_0 = \langle 7-4, -3-2, -2-(-1)\rangle = \langle 3, -5, -1\rangle.$$

Thus, the line can be written

$$\langle x, y, z\rangle = \langle 4,2,-1\rangle + t\langle 3, -5, -1\rangle = \langle 4+3t, 2-5t, -1-t\rangle, \qquad \text{for } 0 \le t \le 1.$$

The bounds are such that $t = 0$ gives $P_0 = (4, 2, -1)$ and $t = 1$ gives $P_1 = (7, -3, -2)$. Since there is a direction implied by increasing t, this is called a *directed line segment*.

When a line segment between two points is constructed in this manner, the bounds on t are always $0 \le t \le 1$.

Example 3: Find the parametric equation of the line segment connecting $P_0 = (4, 2, -1)$ and $P_1 = (7, -3, -2)$ such that $t = 0$ gives P_0 and that $t = 5$ gives P_1.

Solution: The difference in t-values is $b - a = 5 - 0 = 5$. Thus, the line segment is

$$\langle x, y, z \rangle = \langle 4,2,-1 \rangle + \frac{t}{5}\langle 3,-5,-1 \rangle$$

$$= \langle 4,2,-1 \rangle + t\left\langle \frac{3}{5}, -1, -\frac{1}{5} \right\rangle$$

$$= \left\langle 4 + \frac{3}{5}t, 2 - t, -1 - \frac{1}{5}t \right\rangle, \qquad \text{for } 0 \le t \le 5.$$

Note that $t = 0$ gives P_0 and that $t = 5$ gives P_1.

Example 4: Let L_1: $\langle x, y, z\rangle = \langle 1,2,5\rangle + t\langle 2,4,-3\rangle$ and L_2: $\langle x, y, z\rangle = \langle 6,1,-2\rangle + s\langle 4,8,-6\rangle$ be two lines defined parametrically. (a) Are lines L_1 and L_2 parallel? (b) Are lines L_1 and L_2 the same line?

Solution:

(a) The direction vector for line L_1 is $\mathbf{v}_1 = \langle 2,4,-3\rangle$ and the direction vector for line L_2 is $\mathbf{v}_2 = \langle 4,8,-6\rangle$. Since $\mathbf{v}_2 = 2\mathbf{v}_1$ (or equivalently, $\mathbf{v}_1 = \frac{1}{2}\mathbf{v}_2$), the two vectors are parallel. Thus, so are the lines.

(b) Choose a point from one line and show that the other line passes through it. In this example, choose point $P_0 = (1,2,5)$ from line L_1. Does L_2 pass through $P_0 = (1,2,5)$? We substitute the coordinates in P_0 for x, y and z, and attempt to solve for a unique value of s that would indicate line L_2 passes through a point in line L_1:

$$1 = 6 + 4s, \qquad 2 = 1 + 8s, \qquad 5 = -2 - 6s.$$

From the first equation, we get $s = -5/4$, but from the second equation, we get $s = 1/8$. Since s is not unique, we conclude it is impossible that L_2 passes through $P_0 = (1,2,5)$. Thus, lines L_1 and L_2 represent two different parallel lines.

Example 5: Find the point of intersection of lines L_1: $\langle x, y, z\rangle = \langle 1,2,-1\rangle + t\langle 2,-3,4\rangle$ and L_2: $\langle x, y, z\rangle = \langle 1,8,9\rangle + s\langle 4,-12,-2\rangle$.

Solution: The direction vectors are $\mathbf{v}_1 = \langle 2,-3,4\rangle$ and $\mathbf{v}_2 = \langle 4,-12,-2\rangle$. Since they are not scalar multiples of one another, the two lines are not parallel. To see if they intersect, we set the equations for *x* equal to one another, and for *y*, and for *z*:

$$\begin{aligned} x: &\quad 1 + 2t = 1 + 4s \\ y: &\quad 2 - 3t = 8 - 12s \\ z: &\quad -1 + 4t = 9 - 2s. \end{aligned}$$

Simplifying, we have a system of two variables in three equations:

$$\begin{aligned} 2t - 4s &= 0 \\ -3t + 12s &= 6 \\ 4t + 2s &= 10. \end{aligned}$$

One of two things happens: either we find a solution in s and t, in which case there is an intersection point, or we do not find a solution in s and t, in which case there is no intersection point. From the first two equations, we solve a system:

$$\begin{aligned} 2t - 4s &= 0 \\ -3t + 12s &= 6 \end{aligned} \xrightarrow{\left(\substack{\text{Multiply} \\ \text{top row by 3}}\right)} \begin{aligned} 3(2t - 4s) &= 0 \\ -3t + 12s &= 6 \end{aligned} \xrightarrow{\left(\substack{\text{Distribute} \\ \text{then add}}\right)} \begin{aligned} 6t - 12s &= 0 \\ -3t + 12s &= 6 \end{aligned} \rightarrow \quad 3t = 6.$$

Thus, we have $t = 2$, and back-substituting, we have $s = 1$. Does this solve the third equation? We substitute and simplify:

$$4(2) + 2(1) = 8 + 2 = 10.$$

We get a true statement. We were able to show that when $t = 2$, we generate the point $(5, -4, 7)$ on line L_1, and when $s = 1$, we generate the same point $(5, -4, 7)$ on line L_2. Thus, the two lines intersect at this point.

Planes

Given a point $P_0 = (x_0, y_0, z_0)$ and a vector $\mathbf{n} = \langle a, b, c \rangle$, a **plane** that passes through P_0 and is normal (orthogonal) to **n** has the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

which simplifies to $ax + by + cz = d$, where $d = ax_0 + by_0 + cz_0$.

Example 6: State a vector that is normal to the plane $2x - 4y + 3z = 12$.

Solution: There are infinitely-many possible vectors. One is $\langle 2, -4, 3 \rangle$, which is found by reading the coefficients of x, y and z. Any non-zero multiple of $\langle 2, -4, 3 \rangle$ is also a vector normal to the plane.

Example 7: Find the equation of the plane passing through $P_0 = (-3,9,1)$ and normal to $\mathbf{n} = \langle 7,3,-5 \rangle$.

Solution: The plane has the equation

$$7\big(x-(-3)\big)+3(y-9)-5(z-1)=0.$$

Simplifying, we have

$$7(x+3)+3(y-9)-5(z-1)=0$$

$$7x+21+3y-27-5z+5=0$$

$$7x+3y-5z=1.$$

Example 8: Find the point of intersection of the line $\langle x, y, z\rangle = \langle 4 - t, 2 + 3t, 3 - 5t\rangle$ and the plane $6x + 2y - 3z = 79$.

Solution: Substitute the equations for x, y and z into the plane, and solve for t:

$$\begin{aligned} 6(4 - t) + 2(2 + 3t) - 3(3 - 5t) &= 79 \\ 24 - 6t + 4 + 6t - 9 + 15t &= 79 \\ 15t &= 60 \\ t &= 4\,. \end{aligned}$$

Now substitute $t = 4$ into the equations for the line:

$$\langle x, y, z\rangle = \langle 4 - (4), 2 + 3(4), 3 - 5(4)\rangle = \langle 0,14,-17\rangle.$$

This is a vector, but if referenced from the origin, its head lies at the point $(0,14,-17)$.

Example 9: Find the equation of the plane passing through the points $A = (1,3,4)$, $B = (-3,2,6)$ and $C = (1,0,-6)$.

Solution: From the three points, form two vectors. For example, vectors **AB** and **AC**:

$$\mathbf{AB} = \langle -4,-1,2 \rangle,\ \ \mathbf{AC} = \langle 0,-3,-10 \rangle$$

Next, find a vector **n** normal to **AB** and **AC** by finding the cross product $\mathbf{AB} \times \mathbf{AC}$:

$$\mathbf{n} = \mathbf{AB} \times \mathbf{AC} = \langle 16,-40,12 \rangle$$

Any non-zero multiple of **n** will suffice, so we divide through by 4, getting $\mathbf{n} = \langle 4,-10,3 \rangle$.

Using any one of the three given points, we now find the equation of the plane. We'll use A first, then check our work with B and C. Using $A = (1,3,4)$ and $\mathbf{n} = \langle 4, -10, 3 \rangle$, we have

$$4(x-1) - 10(y-3) + 3(z-4) = 0$$

$$4x - 4 - 10y + 30 + 3z - 12 = 0$$

$$4x - 10y + 3z = -14.$$

Example 10: Find the acute angle formed by the intersection of the planes $x + 3y - 2z = 5$ and $4x - y + 5z = -2$.

Solution: The respective normal vectors of each plane are $\mathbf{n}_1 = \langle 1,3,-2 \rangle$ and $\mathbf{n}_2 = \langle 4,-1,5 \rangle$. The angle between these two planes is the same as the angle between the two normal vectors:

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{-9}{\sqrt{14 \cdot 42}}\right) \approx 111.79 \text{ degrees}$$

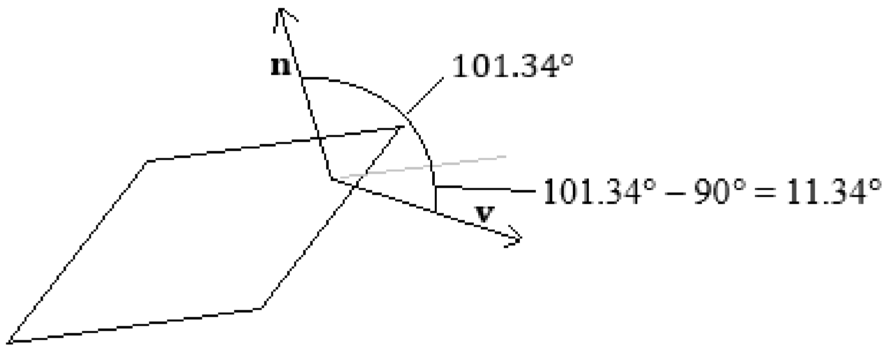
However, planes always intersect at an acute angle (except in the case where they are orthogonal). The preferred answer is the supplement: 180° – 111.79° = 68.21°.

Example 11: Find the acute angle formed by line $\mathbf{v} = \langle 1 + 2t, 3 + t, 5 - 8t \rangle$ and plane $x + 2y + z = 4$.

Solution: The line's direction vector is $\mathbf{v} = \langle 2,1,-8 \rangle$ and the plane's normal vector is $\mathbf{n} = \langle 1,2,1 \rangle$, so the angle between **v** and **n** is

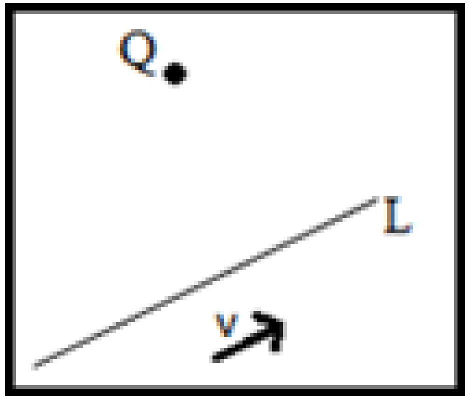
$$\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{n}}{|\mathbf{v}||\mathbf{n}|}\right) = \cos^{-1}\left(\frac{-4}{\sqrt{69 \cdot 6}}\right) \approx 101.34°.$$

In this example, the vector **n** is on one side of the plane, and **v** on the opposite side. Since the angle from **n** to the plane is 90°, the remainder, 101.34° – 90° = 11.34°, is the desired angle.



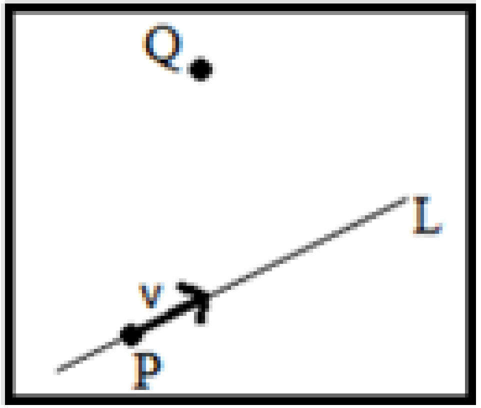
A Pictorial Guide to Finding the Shortest Distance Between a Point and a Line

1.



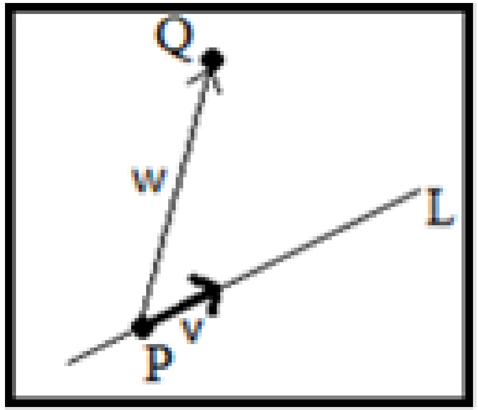
Start with a line *L*, a point *Q* not on *L*, and a direction vector v parallel to *L*.

2.



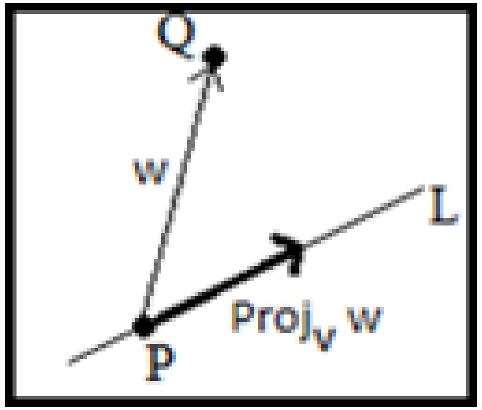
Pick *any* point P on *L*.

3.



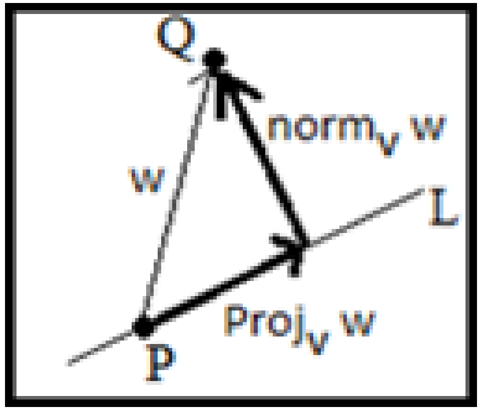
Form a vector w from *P* to *Q*.

4.



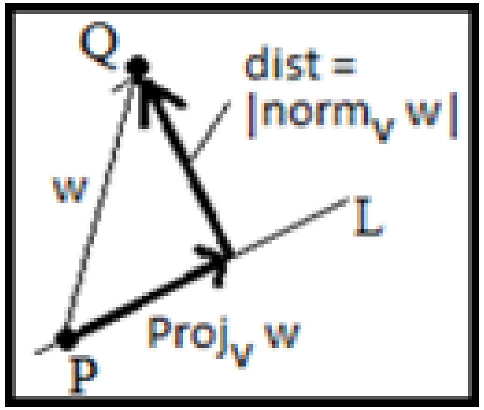
Find the projection of w onto v.

5.



Find a vector normal to the projection, i.e. completing a right triangle.

6.



The distance from *Q* to *L* is the magnitude of this normal vector.

Example 12: Find the shortest distance between the line $\langle x, y, z\rangle = \langle 1,2,-1\rangle + t\langle 2,-3,4\rangle$ and the point $Q = (4,8,3)$.

Solution: Choose any point P on the line. For example, when $t = 0$, we have $P = (1,2,-1)$. Then find the vector from P to Q, which is $\mathbf{w} = \langle 4-1, 8-2, 3-(-1)\rangle = \langle 3,6,4\rangle$. Meanwhile, the directional vector of the line is $\mathbf{v} = \langle 2,-3,4\rangle$. The projection of $\mathbf{w}$ onto $\mathbf{v}$ is:

$$\operatorname{proj}_{\mathbf{v}} \mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{4}{29}\mathbf{v} = \left\langle \frac{8}{29}, -\frac{12}{29}, \frac{16}{29}\right\rangle.$$

The normal vector is found by subtracting $\operatorname{proj}_{\mathbf{v}} \mathbf{w}$ from $\mathbf{w}$:

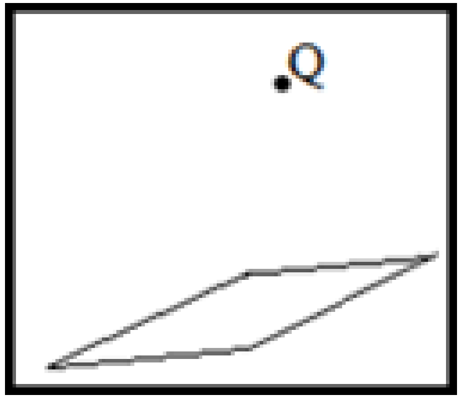
$$\operatorname{norm}_{\mathbf{v}} \mathbf{w} = \mathbf{w} - \operatorname{proj}_{\mathbf{v}} \mathbf{w} = \langle 3,6,4\rangle - \left\langle \frac{8}{29}, -\frac{12}{29}, \frac{16}{29}\right\rangle = \left\langle \frac{79}{29}, \frac{186}{29}, \frac{100}{29}\right\rangle.$$

The magnitude of this normal vector is the distance from the point Q to the line:

$$|\operatorname{norm}_{\mathbf{v}} \mathbf{w}| = \sqrt{\left(\frac{79}{29}\right)^2 + \left(\frac{186}{29}\right)^2 + \left(\frac{100}{29}\right)^2} \approx 7.78 \text{ units}.$$

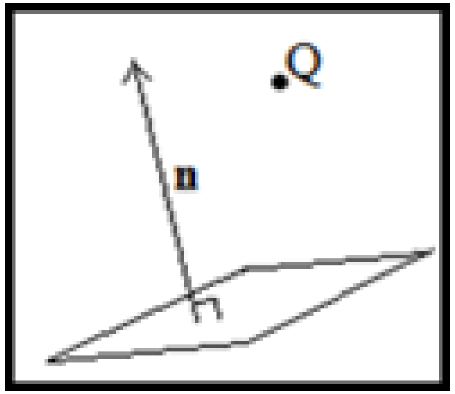
A Pictorial Guide to Finding the Shortest Distance Between a Point and a Plane

1.



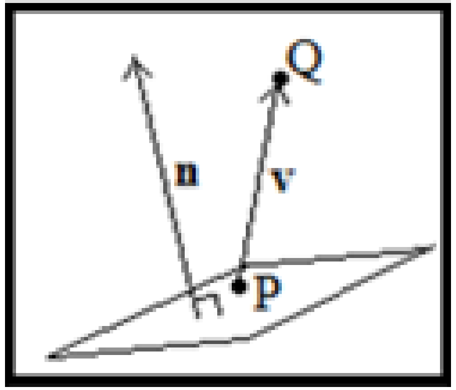
Start with a plane and a point Q not on the plane.

2.



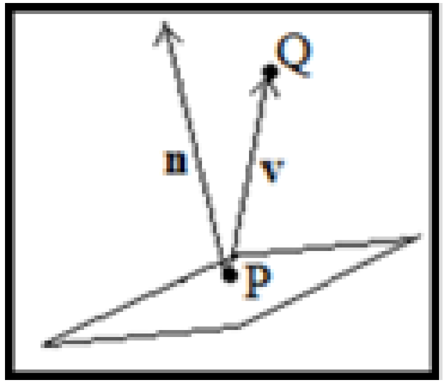
Find $\mathbf{n}$, the normal vector to the plane.

3.



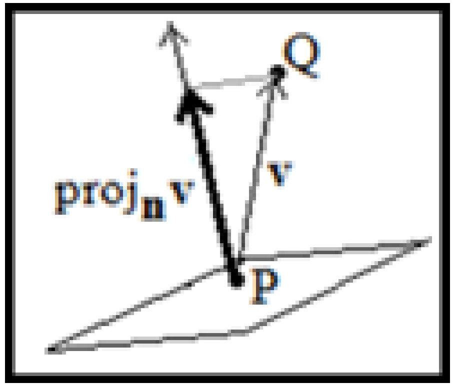
Find *any* point P on the plane, and then form a vector $\mathbf{v}$ from P to Q.

4.



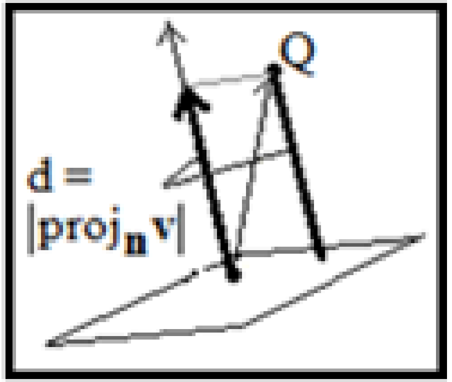
If it helps, you can visualize the vectors $\mathbf{v}$ and $\mathbf{n}$ having a common foot.

5.



Project $\mathbf{v}$ onto $\mathbf{n}$.

6.



The distance from Q to the plane is the magnitude of $\text{proj}_{\mathbf{n}}\,\mathbf{v}$.

Example 13: Find the distance between point $Q = (1,4,3)$ and plane $x - 3y + 2z = 6$.

Solution: Pick any point P in the plane by choosing values for two of the variables and solving for the third. If $x = 0$ and $y = 0$, we get $z = 3$, so a point in the plane is $P = (0,0,3)$. The vector **v** from P to Q is $\mathbf{v} = \langle 1,4,0 \rangle$. The normal vector to the plane is $\mathbf{n} = \langle 1,-3,2 \rangle$. Project **v** onto **n**:

$$\text{proj}_{\mathbf{n}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\mathbf{n} = -\frac{11}{14}\langle 1,-3,2 \rangle.$$

The magnitude of this vector is the distance from $Q = (1,4,3)$ to the plane $x - 3y + 2z = 6$:

$$|\text{proj}_{\mathbf{n}}\mathbf{v}| = \frac{11}{14}\sqrt{1^2 + (-3)^2 + 2^2} = \frac{11}{14}\sqrt{14} \approx 2.94 \text{ units.}$$