Partial Differentiation

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Given a function $z = f(x, y)$. There are two "convenient" directions in which to calculate an instantaneous rate of change: the positive x direction, or the positive y direction.

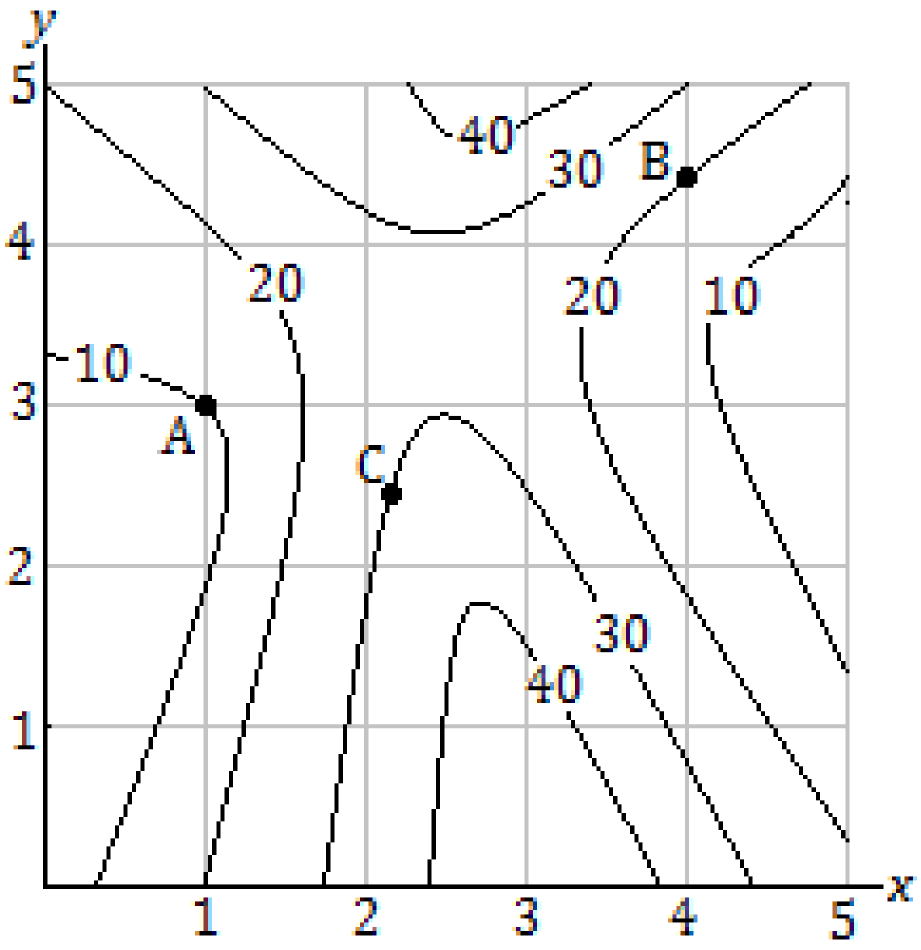
The instantaneous rate of change of f with respect to x is called **the partial derivative of *z* with respect to *x***, and is written

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x}, \qquad \text{or informally as } z_x \text{ or } f_x.$$

Similarly, the instantaneous rate of change of f with respect to y is called **the partial derivative of *z* with respect to *y***, and is written

$$\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial f}{\partial y}, \qquad \text{or informally as } z_y \text{ or } f_y.$$

Example 1: Use the contour map below, representing $z = f(x, y)$. Assume that f is smooth and continuous. Give the sign of $f_x(A)$, $f_y(A)$, $f_x(B)$, $f_y(B)$, $f_x(C)$, and $f_y(C)$.



$f_x(A)$: $+$

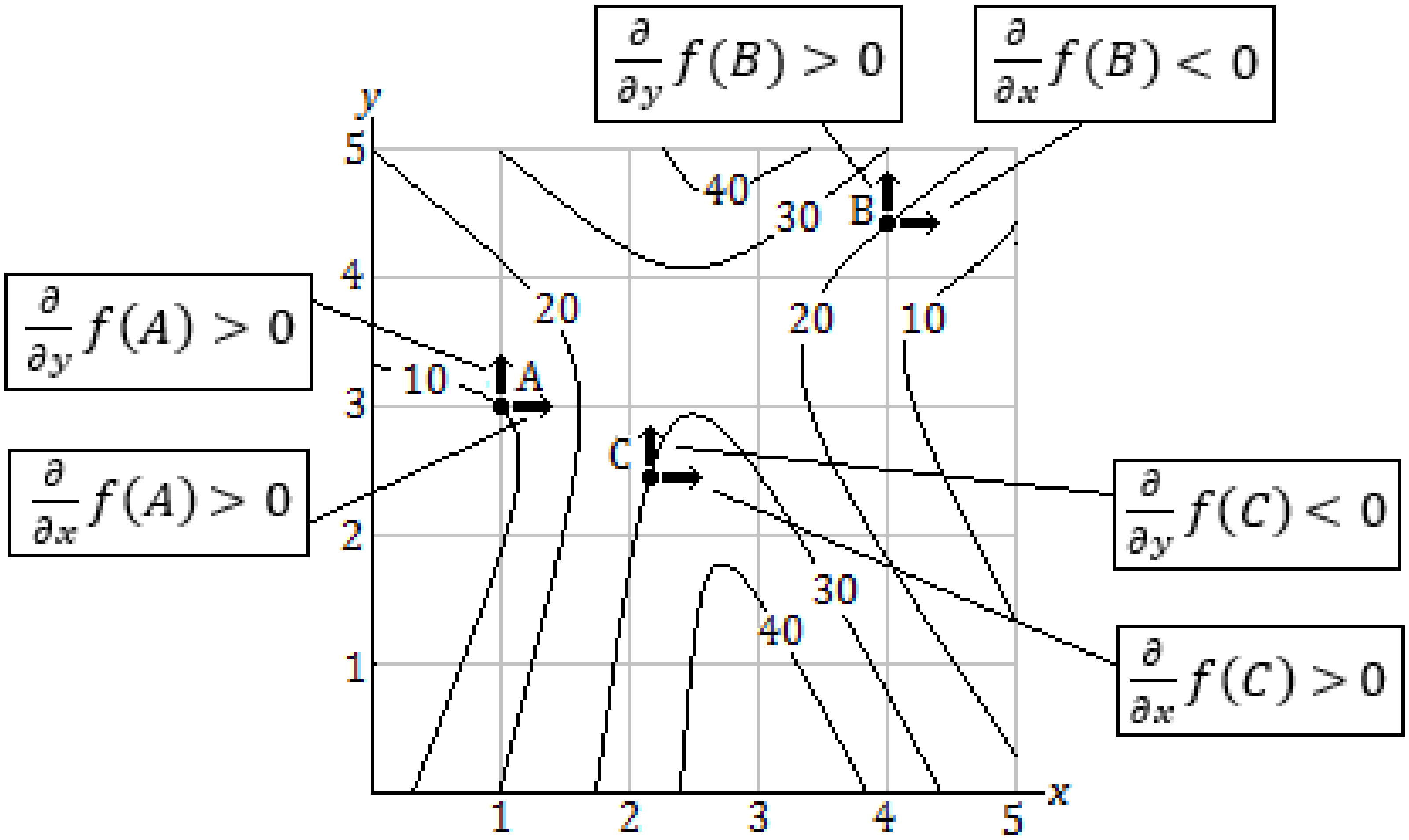
$f_y(A)$: $+$

$f_x(B)$: $-$

$f_y(B)$: $+$

$f_x(C)$: $+$

$f_y(C)$: $-$

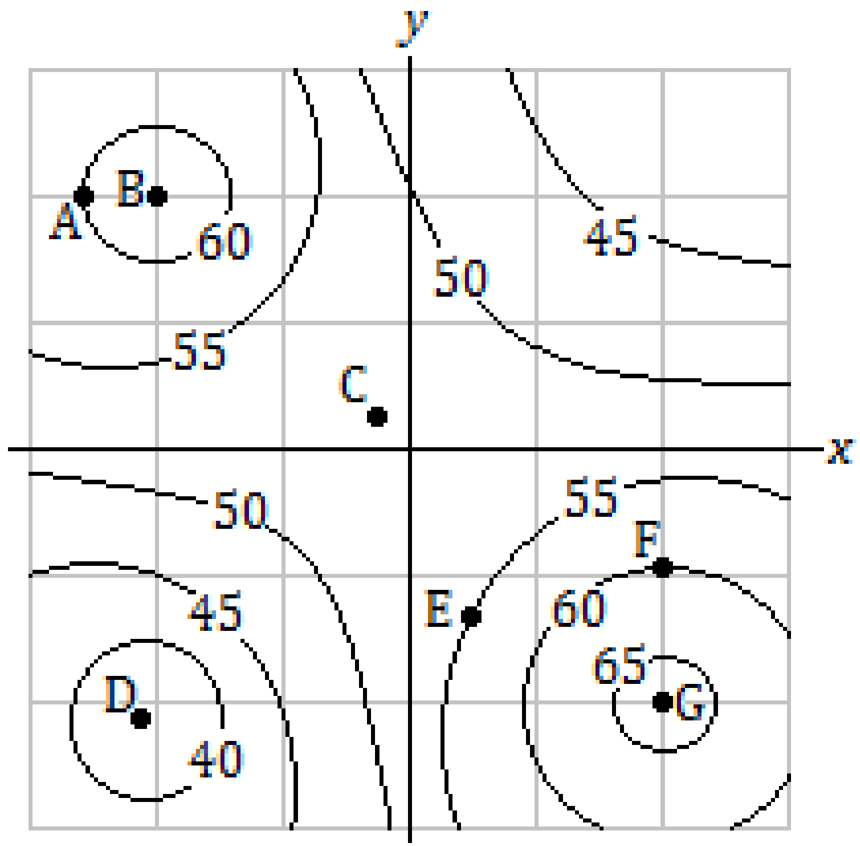


$\frac{\partial}{\partial y} f(B) > 0$
$\frac{\partial}{\partial x} f(B) < 0$
$\frac{\partial}{\partial y} f(A) > 0$
$\frac{\partial}{\partial x} f(A) > 0$
$\frac{\partial}{\partial y} f(C) < 0$
$\frac{\partial}{\partial x} f(C) > 0$
y
x
1
2
3
4
5
1
2
3
4
5
A
B
C
10
20
30
40
30
20
10
30
40

On a contour map representing the surface of a smooth and continuous function f, the values of the partial derivatives of f with respect to x and with respect to y are 0 at all minimum, maximum and saddle points.

If movement in the x or y direction happens to be tangential to the contour at a point, then the value of the partial derivative of f with respect to the x or y direction is 0. That is, tangential movement along a level curve always means no change in z.

Example 2: The contour map below represents the surface of a smooth and continuous function $z = g(x, y)$. Assume that points B, C, D and G are minimum, maximum or saddle points. State the sign (positive, negative or zero) of the partial derivative of g with respect to x and with respect to y, at each of the points A through G.



$\frac{\partial}{\partial x} g(A) > 0$ and $\frac{\partial}{\partial y} g(A) = 0$.

$\frac{\partial}{\partial x} g(B) = 0$ and $\frac{\partial}{\partial y} g(B) = 0$. B is a local maximum.

$\frac{\partial}{\partial x} g(C) = 0$ and $\frac{\partial}{\partial y} g(C) = 0$. C is a saddle point.

$\frac{\partial}{\partial x} g(D) = 0$ and $\frac{\partial}{\partial y} g(D) = 0$. D is a local minimum.

$\frac{\partial}{\partial x} g(E) > 0$ and $\frac{\partial}{\partial y} g(E) < 0$.

$\frac{\partial}{\partial x} g(F) = 0$ and $\frac{\partial}{\partial y} g(F) < 0$.

$\frac{\partial}{\partial x} g(G) = 0$ and $\frac{\partial}{\partial y} g(G) = 0$. G is a local maximum.

Example 3: Let $z = f(x, y) = x^2y + 3x^3y^4 + 2x - 4y$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: When finding $\frac{\partial z}{\partial x}$, treat the *y* as a constant. If it is in a term by itself, then the whole term is treated as a constant. If it is connected to *x* through multiplication, then it is treated as a coefficient. Thus, we have

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2y + 3x^3y^4 + 2x - 4y) = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial x}(3x^3y^4) + \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial x}(4y)$$
$$= (2x)y + 3(3x^2)y^4 + 2(1) - 0$$
$$= 2xy + 9x^2y^4 + 2.$$

Similarly, to find $\frac{\partial z}{\partial y}$, we treat *x* as a constant or a coefficient:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2y + 3x^3y^4 + 2x - 4y) = \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial y}(3x^3y^4) + \frac{\partial}{\partial y}(2x) - \frac{\partial}{\partial y}(4y)$$
$$= x^2(1) + 3x^3(4y^3) + 0 - 4(1)$$
$$= x^2 + 12x^3y^3 - 4.$$

Example 4: Let $z = x^3 \sin(x^2 y^3)$. Find z_x and z_y.

Solution: For z_x, note that x is present in two factors attached by multiplication. Thus, we use the Product Rule of differentiation and the Chain Rule:

$$z_x = x^3(\cos(x^2y^3)\,2xy^3) + 3x^2\sin(x^2y^3) = 2x^4y^3\cos(x^2y^3) + 3x^2\sin(x^2y^3)\,.$$

For z_y, we do not need the Product Rule, treating the x^3 as a coefficient of the sine operator. However, we do need the Chain Rule:

$$z_y = x^3\cos(x^2y^3)\,x^2(3y^2) = 3x^5y^2\cos(x^2y^3)\,.$$

Higher-Order Partial Derivatives & Clairaut's Theorem

Suppose $z = f(x, y)$ is given. There are two first partial derivatives, $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$.

Each partial derivative is itself a function of two variables. Thus, each has two partial derivatives of its own. For example, $f_x(x, y)$ has two partial derivatives:

$$(f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \quad \text{and} \quad (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} .$$

Similarly, $f_y(x, y)$ has two partial derivatives:

$$\left(f_y\right)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \quad \text{and} \quad \left(f_y\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} .$$

Usually, second derivatives are noted by using subscripts without parentheses. Thus,

$$f_{xx} = (f_x)_x, \qquad f_{yy} = \left(f_y\right)_y, \qquad f_{xy} = (f_x)_y \ \text{ and } \ f_{yx} = \left(f_y\right)_x.$$

Second derivatives such as f_{xx} and f_{yy} are informally called *homogeneous* second derivatives, while f_{xy} and f_{yx} are called *mixed* second derivatives.

Under "typical" circumstances, *e.g.* the function f being smooth and continuous, and twice-differentiable over its relevant domain, the mixed second derivatives will be equal:

$$f_{xy} = f_{yx} \quad \text{(Clairaut's Theorem).}$$

As one might expect, second derivatives of a smooth and continuous function offer insight to the concavity of the function.

Example 5: Given $z = f(x, y) = x^2y + 3x^3y^4 + 2x - 4y$. Find f_{xx}, f_{yy}, f_{xy} and f_{yx}.

Solution: From a previous example, we found the two first partial derivatives:

$$f_x(x, y) = 2xy + 9x^2y^4 + 2 \quad \text{and} \quad f_y(x, y) = x^2 + 12x^3y^3 - 4.$$

Thus, we have

$$f_{xx} = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial}{\partial x}(2xy + 9x^2y^4 + 2) = 2y + 18xy^4$$

and

$$f_{yy} = \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial}{\partial y}(x^2 + 12x^3y^3 - 4) = 36x^3y^2.$$

Furthermore, we have

$$f_{xy} = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial y}(2xy + 9x^2y^4 + 2) = 2x + 36x^2y^3$$

and

$$f_{yx} = \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial}{\partial x}(x^2 + 12x^3y^3 - 4) = 2x + 36x^2y^3.$$

Note that $f_{xy} = f_{yx}$.

Example 6: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, where $f(x,y) = \int_x^y 3t^2\,dt$.

Solution: Defining functions as integrals is not uncommon. In this case, we can antidifferentiate the integrand, and evaluate at the limits of integration:

$$f(x,y) = \int_x^y 3t^2\,dt = [t^3]_x^y = y^3 - x^3.$$

Taking partial derivatives, we have,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y^3 - x^3) = -3x^2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^3 - x^3) = 3y^2.$$

Note that the results look similar to the original integrand. Was it necessary to do the antidifferentiation step? See the next example.

Example 7: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, where $f(x, y) = \int_x^y \sqrt{t^4 - 2t + 7}\, dt$.

Solution: Repeating the steps of the previous example leads to what appears to be an impossible step: the integrand does not antidifferentiate "conveniently" into common elementary functions. For now, define $H(t)$ to be the antiderivative of $\sqrt{t^4 - 2t + 7}$. We cannot determine $H(t)$, but we know its derivative is $\sqrt{t^4 - 2t + 7}$. Thus, we have

$$f(x, y) = \int_x^y \sqrt{t^4 - 2t + 7}\, dt = [H(t)]_x^y = H(y) - H(x).$$

Thus,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\big(H(y) - H(x)\big) = -\sqrt{x^4 - 2x + 7},$$

where $\frac{\partial}{\partial x}\big(H(y)\big) = 0$ and where $\frac{\partial}{\partial x}\big(H(x)\big)$ is the derivative of $H(t)$ with x in place of t. In a similar manner,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big(H(y) - H(x)\big) = \sqrt{y^4 - 2y + 7}.$$