Projectile Motion

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On Earth, the gravitational acceleration constant is -9.8 meters per second per second (or $\mathrm{m/s^2}$). If we superimpose an *xy*-axis system with the positive *y* axis being "up", then **acceleration** can be written as a vector-valued function,

$$\mathbf{a}(t) = \langle 0, -9.8 \rangle.$$

The *x*-component of acceleration is 0, since falling bodies will not accelerate horizontally due to gravity. Note that both components are constants.

Integrating acceleration, we get **velocity**, which is also a vector-valued function:

$$\mathbf{v}(t) = \int \mathbf{a}(t)\ dt = \int \langle 0, -9.8 \rangle\ dt = \langle 0, -9.8t \rangle + \langle v_x, v_y \rangle = \langle v_x, -9.8t + v_y \rangle.$$

Here, $\langle v_x, v_y \rangle$ are constants of integration and represents the initial velocity in the *x*-direction and in the *y*-direction, respectively. **Speed** is the magnitude of velocity. Speed is a scalar value.

Integrating velocity, we get **displacement** (or **position**), also a vector-valued function:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \langle v_x, -9.8t + v_y \rangle\ dt = \langle v_x t + r_x, -4.9t^2 + v_y t + r_y \rangle.$$

Similar to above, $\langle r_x, r_y \rangle$ represent the initial position of the object in the *x*-direction and *y*-direction respectively. The placement of the origin is arbitrary but is usually done so that $r_x = 0$ since it is almost always practical to assume no initial horizontal distance.

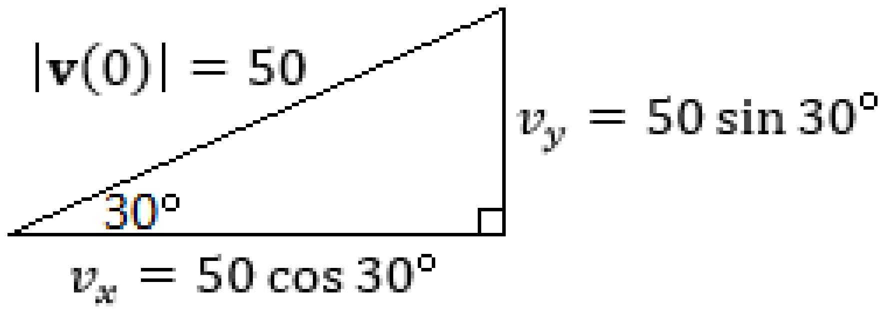
Example 1: A ball is propelled off a 100 m tall cliff at an initial speed of 50 meters per second at an angle of 30 degrees above the horizontal.

a) Find the maximum height of the ball.
b) Find the range of the ball when hits the ground for the first time.
c) How fast is the ball travelling when it impacts the ground for the first time?
d) At what angle does the ball impact the ground for the first time?

Solution: Starting with acceleration, $\mathbf{a}(t) = \langle 0, -9.8 \rangle$, integrate to obtain

$$\mathbf{v}(t) = \langle v_x, -9.8t + v_y \rangle.$$

To find v_x and v_y, note that its initial speed is $|\mathbf{v}(0)| = 50$ at an angle of 30 degrees. This suggests a right triangle, in which $|\mathbf{v}(0)| = 50$ is the hypotenuse, and v_x and v_y are the horizontal and vertical legs, respectively.



Thus, $v_x = 50 \cos 30 = 25\sqrt{3} \approx 43.301$, and $v_y = 50 \sin 30 = 25$, and the velocity vector is

$$\mathbf{v}(t) = \langle 43.301, -9.8t + 25 \rangle.$$

Integrating **v**, we obtain the displacement vector

$$\mathbf{r}(t) = \langle 43.301t + r_x, -4.9t^2 + 25t + r_y \rangle.$$

Since the ball was thrown off the top of a cliff, set the origin at the base of the cliff, so that $r_x = 0$ and $r_y = 100$. Therefore, the displacement vector is

$$\mathbf{r}(t) = \langle 43.301t, -4.9t^2 + 25t + 100 \rangle.$$

We now have sufficient information to answer the questions. In all cases, assume $t \geq 0$.

a) The ball reaches its maximum height when the *y*-component of velocity is 0 since the ball's vertical velocity is 0 (momentarily stops) when it reaches it maximum height.

Thus, solve $-9.8t + 25 = 0$ to find *when* the ball has reached its maximum height.

This happens at $t = 25/9.8 = 2.551$ seconds.

Substituting this into the *y*-component of displacement, we now determine the height of the ball at this time:

$$\text{Maximum height} = -4.9(2.551)^2 + 25(2.551) + 100 = 131.888 \text{ meters.}$$

b) The ball impacts the ground when the y-component of displacement is 0. We use the quadratic formula to find the roots of $-4.9t^2 + 25t + 100 = 0$:

$$t = \frac{-25 \pm \sqrt{25^2 - 4(-4.9)(100)}}{2(-4.9)} \quad \rightarrow \quad t \approx -2.637 \text{ or } 7.739.$$

The negative result is ignored. The ball lands at $t \approx 7.739$ seconds.

The range in which the ball travelled from the base is found by evaluating the x-component of displacement by this t:

$$\text{Range} = 43.301(7.739) = 335.106 \text{ m}.$$

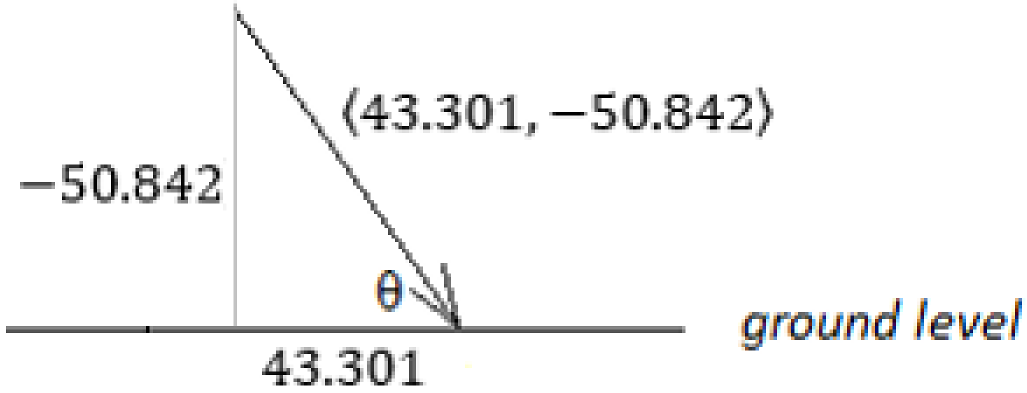
c) We find the velocity at t = 7.739 seconds:

$$\mathbf{v}(7.739) = \langle 43.301, -9.8(7.739) + 25 \rangle = \langle 43.301, -50.842 \rangle.$$

The impact speed of the ball is the magnitude of this vector:

$$\text{Speed} = |\langle 43.301, -50.842 \rangle| = \sqrt{43.301^2 + (-50.842)^2} \approx 66.702 \text{ m/s}.$$

d) To find the angle at which the ball impacts the ground, sketch a diagram to be sure that the components of velocity are properly in place:



The angle of impact is $\theta = \tan^{-1}\frac{-50.842}{43.301} = -49.6$ degrees. The negatives can be ignored, so that the impact angle is 49.6 degrees.

Example 2: A golf ball is hit from the ground by an astronaut on a distant planet. It reaches a maximum height of 50 meters after 4 seconds of flight. What is the gravitational constant on this planet?

Solution: We start with $\mathbf{a}(t) = \langle 0, -a \rangle$, then develop $\mathbf{v}(t)$ and $\mathbf{r}(t)$:

$$\mathbf{v}(t) = \langle v_x, -at + v_y \rangle \quad \text{and} \quad \mathbf{r}(t) = \langle v_x t + r_x, -\frac{a}{2}t^2 + v_y t + r_y \rangle.$$

Assume that the initial position is $\langle r_x, r_y \rangle = \langle 0,0 \rangle$. Thus, position is given by $\mathbf{r}(t) = \langle v_x t, -\frac{a}{2}t^2 + v_y t \rangle, t \geq 0$.

When the ball reaches the maximum height, its vertical component of velocity is 0, while the vertical component of position is 50. This creates a pair of equations:

$$-at + v_y = 0$$

$$-\frac{a}{2}t^2 + v_y t = 50.$$

When $t = 4$, we obtain $-4a + v_y = 0$ and $-\frac{a}{2}(4)^2 + 4v_y = 50$. From the first equation, we have

$$v_y = 4a,$$

and from the second equation, we have

$$-8a + 4(4a) = 50.$$

Solving for a, we obtain

$$-8a + 16a = 50$$

$$8a = 50$$

$$a = \frac{50}{8} = 6.25.$$

Evidently, objects on this planet fall at a rate of 6.25 meters per second^2.