Integration in Spherical Coordinates

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Example 1: Evaluate $\iiint_R \sqrt{x^2+y^2+z^2}\, dV$, where R is a hemisphere of radius 5, centered at the origin and above the *xy*-plane.

Solution: In rectangular coordinates, the triple integral is

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\ dy\ dx\,.$$

In spherical coordinates, the integrand is rewritten as $\sqrt{x^2+y^2+z^2}=\sqrt{\rho^2}=\rho$, then multiplied by the Jacobian $\rho^2\sin\phi$. This same integral in spherical coordinates is

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{5}(\rho)\,\rho^2\sin\phi\ d\rho\ d\phi\ d\theta=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{5}\rho^3\sin\phi\ d\rho\ d\phi\ d\theta\,.$$

Because the integrand is held by multiplication and the bounds of integration are constant, the triple integral can be performed as three separate single-variable integrals:

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^5 \rho^3 \sin\phi \; d\rho \; d\phi \; d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi/2} \sin\phi \, d\phi\right)\left(\int_0^5 \rho^3 \, d\rho\right)$$

$$= (2\pi)\left([-\cos\phi]_0^{\pi/2}\right)\left(\left[\frac{1}{4}\rho^4\right]_0^5\right)$$

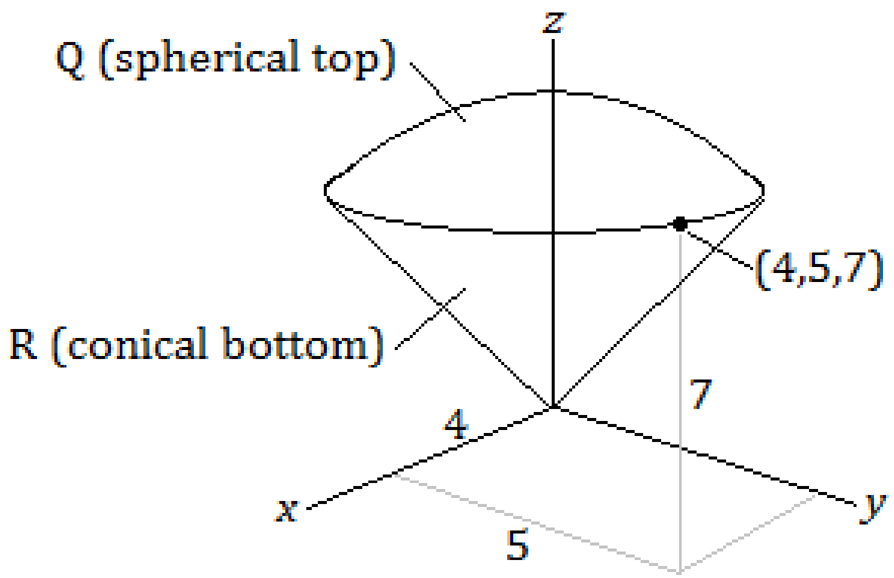
$$= 2\pi\left(-\cos\frac{\pi}{2} - (-\cos 0)\right)\left(\frac{1}{4}(5)^4 - 0\right)$$

$$= 2\pi(0+1)\left(\frac{625}{4}\right)$$

$$= \frac{625}{2}\pi$$

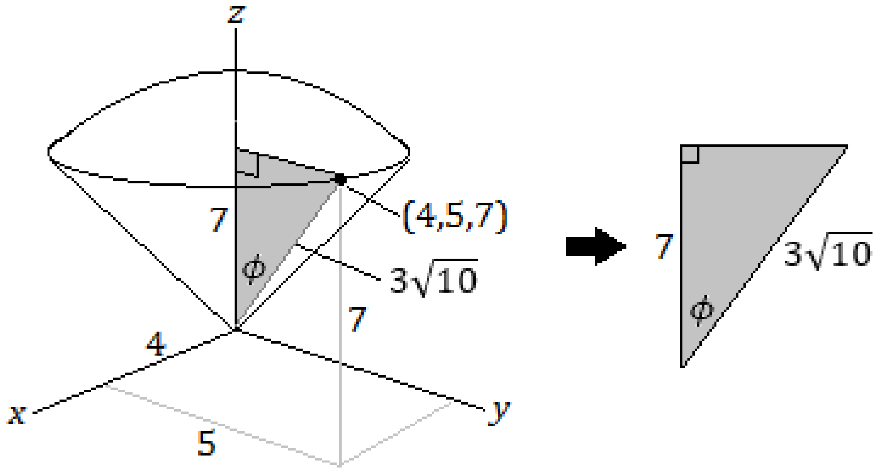
Example 41.3: Let Q be a sphere centered at the origin, and R be a cone whose vertex is at the origin and opens in the positive z direction. The solid S bounded inside the cone and the sphere is called a *spherical sector*. Suppose the point (4,5,7) in rectangular coordinates lies on the "lip", where the sphere and the cone intersect. Find the volume of S.

Solution: Determine bounds for ρ, θ and ϕ by sketching the solid and the point on its rim:



The distance from (0,0,0) to (4,5,7) is $\sqrt{4^2+5^2+7^2}=\sqrt{90}=3\sqrt{10}$. Since the solid includes the origin, the bounds of ρ are $0 \le \rho \le 3\sqrt{10}$.

The solid includes the positive *z*-axis, so the lower bound for ϕ is 0. The upper bound is found by observing a right triangle with the adjacent leg on the *z*-axis, and the hypotenuse corresponding to a line from the origin to the point (4,5,7). From this, we see that for an upper bound, we have $\phi = \arccos\left(\frac{7}{3\sqrt{10}}\right)$.



The solid encircles the *z*-axis. The bounds of θ are $0 \le \theta \le 2\pi$.

The volume integral in spherical coordinates is

$$\int_0^{2\pi}\int_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)}\int_0^{3\sqrt{10}} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta\,.$$

The integrand is held by multiplication and the bounds of integration are constant, so the triple integral can be performed as three separate single-variable integrals:

$$\int_0^{2\pi}\int_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)}\int_0^{3\sqrt{10}} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)} \sin\phi \; d\phi\right)\left(\int_0^{3\sqrt{10}} \rho^2 \; d\rho\right)$$

$$= (2\pi)\left([-\cos\phi]_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)}\right)\left(\left[\frac{1}{3}\rho^3\right]_0^{3\sqrt{10}}\right)$$

$$= 2\pi\left(-\cos\left(\arccos\left(\frac{7}{3\sqrt{10}}\right)\right) - (-\cos 0)\right)\left(\frac{1}{3}\left(3\sqrt{10}\right)^3\right)$$

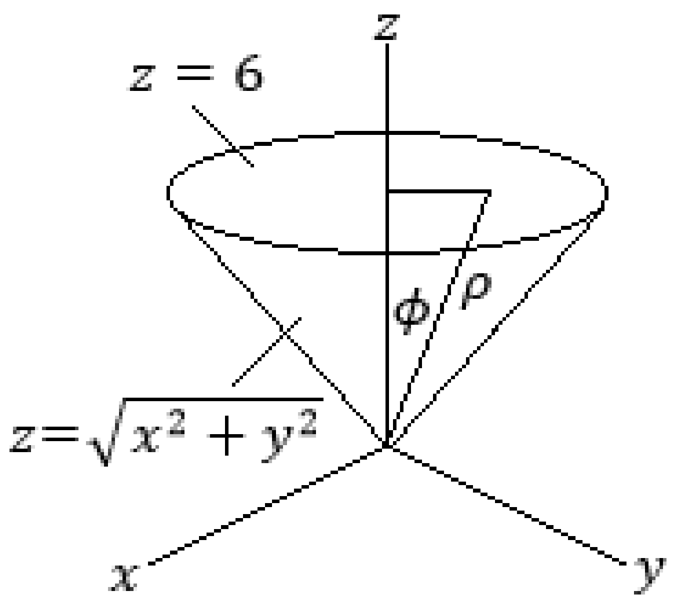
$$= 2\pi\left(1 - \frac{7}{3\sqrt{10}}\right)\left(9(10)^{3/2}\right)$$

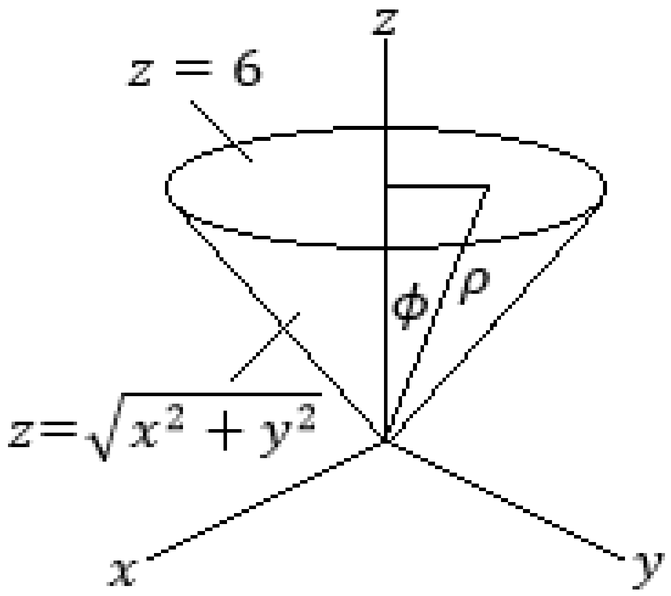
$$= 18\pi(10)^{3/2}\left(1 - \frac{7}{3\sqrt{10}}\right) \approx 468.76 \text{ cubic units}$$

In this example, the bounds are not all constant:

Example 3: Use spherical coordinates to find the volume contained within the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 6$.

Solution: First, observe that the solid is not a spherical sector as in the previous example. The value of ρ will vary as a function of ϕ.





The "sweep" angle θ encompasses a full counter-clockwise rotation around the *xy*-plane from the positive *x*-axis back to the positive *x*-axis, so that $0 \leq \theta \leq 2\pi$.

The "lean" angle ϕ varies from 0 (the positive *z*-axis) to $\frac{\pi}{4}$ (the side of the cone, which is 45 degrees from both the positive *x*-axis and the positive *y*-axis).

For the plane $z = 6$, substitute $z = \rho\cos\phi$, getting $\rho\cos\phi = 6$.

Solving for ρ gives $\rho = 6/\cos\phi = 6\sec\phi$. Since the object is a solid and includes the origin, the lower bound for ρ is 0, while the upper bound is the plane, so that the bounds for ρ are $0 \leq \rho \leq 6\sec\phi$. Thus, the volume integral is

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{6\sec\phi} \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta\,.$$

The inner-most integral is integrated with respect to ρ:

$$\int_0^{6\sec\phi} \rho^2 \sin\phi \; d\rho = \sin\phi \left[\frac{\rho^3}{3}\right]_0^{6\sec\phi}$$

$$= \sin\phi\left(\frac{216\sec^3\phi}{3}\right)$$

$$= 72\sin\phi\cos^{-3}\phi\,.$$

This is now integrated with respect to ϕ.

Note that $72\sin\phi\cos^{-3}\phi$ can be antidifferentiated by a *u*-*du* substitution, where $u = \cos\phi$ so that $du = -\sin\phi \; d\phi$. This results in a power-rule form, $\int(-72u^{-3})\,du = 36u^{-2}$:

$$\int_0^{\pi/4} 72\sin\phi\cos^{-3}\phi \; d\phi = [36\cos^{-2}\phi]_0^{\pi/4}$$

$$= 36\left(\frac{\sqrt{2}}{2}\right)^{-2} - (36(1)^{-2})$$

$$= 36(2) - 36$$

$$= 36\,.$$

Lastly, we integrate with respect to θ:

$$\int_0^{2\pi} (36)\,d\theta = 36(2\pi) = 72\pi \text{ cubic units.}$$