Stokes Theorem

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Recall that Green's Theorem allows us to find the work (as a line integral) performed on a particle around a simple closed loop path C by evaluating a double integral over the interior R that is bounded by the loop:

$$\text{Green's Theorem:} \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(N_x - M_y\right) dA \,.$$

Green's Theorem is restricted to closed loop paths in R^2. What about a closed loop path in R^3? For such paths, we use **Stokes Theorem**, which extends Green's Theorem into R^3.

If $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle$ is a vector field and S is a simple oriented surface in R^3 with a boundary C, then Stokes Theorem is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS \,.$$

Where curl **F** is defined by

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} = \left(P_y - N_z\right)\mathbf{i} - \left(P_x - M_z\right)\mathbf{j} + \left(N_x - M_y\right)\mathbf{k} = \langle P_y - N_z,\ M_z - P_x,\ N_x - M_y\rangle.$$

The integral $\iint_S$ (curl $\mathbf{F}$) $\cdot\, \mathbf{n}\, dS$ needs to be expanded so that it can be useful. Suppose for now that the surface S is defined by $z = f(x, y)$. From this, we have that $\mathbf{n}$ is a normal vector to f by

$$\mathbf{n} = \frac{\langle f_x, f_y, -1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}} \text{ or } \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

Usually, "up" is positive z, so we usually choose this version of **n**.

Also, recall that $dS = \sqrt{f_x^2 + f_y^2 + 1}\, dA$. Thus, making substitutions, we have

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (\text{curl } \mathbf{F}) \cdot \frac{\langle -f_x, -f_y, 1\rangle}{\sqrt{f_x^2 + f_y^2 + 1}} \sqrt{f_x^2 + f_y^2 + 1}\, dA.$$

This simplifies to

$$\iint_R (\text{curl } \mathbf{F}) \cdot \langle -f_x, -f_y, 1\rangle\, dA.$$

The usual routine is:

You will be asked to find the value of a line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ around a simple loop path C in R^3. Path C may be stated explicitly, or may be implied by some surface S given by $z = f(x, y)$. You will also be given the vector field $\mathbf{F} = \langle M, N, P \rangle$.

1. Find curl **F**. For now, it will be in terms of *x*, *y* and *z*.
2. Determine $\langle -f_x, -f_y, 1 \rangle$ or $\langle f_x, f_y, -1 \rangle$, depending on the context. Usually, the first version is used because we can always declare that positive *z* is "up".
3. Find $(\text{curl } \mathbf{F}) \cdot \langle -f_x, -f_y, 1 \rangle$. If variable *z* remains, substitute with $z = f(x, y)$. You now have an expression in terms of *x* and *y*.
4. Determine the region of integration *R*, which will be the footprint cast by *S* onto the *xy*-plane.
5. Integrate the result in step (3) over region *R*.

Normal adjustments would be made, *e.g.* if the surface was stated as $x = f(y, z)$.

Example 1: Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xy, x + y + z, x^2 \rangle$ and C is a circle of radius 1, centered at the origin, in the *xy*-plane, traversed counterclockwise where "up" is the positive *z* direction.

Solution: No surface *S* is specified, just a boundary path *C*. So let's try a couple different surfaces that have *C* as its boundary. First, we will let *S* be the interior of the circle in the *xy*-plane. That is, $z = f(x, y) = 0$. Thus, $\mathbf{n} = \langle 0,0,1 \rangle$.

Next, we find curl **F**:

$$\text{curl } \mathbf{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle = \langle -1, -2x, 1 - x \rangle.$$

Thus, $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = 1 - x$. This is integrated over the region inside the circle of radius 1, centered at the origin. We use polar coordinates, where $x = r\cos\theta$:

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_S (1 - x) \, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 - r\cos\theta) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (r - r^2 \cos\theta) \, dr \, d\theta \, .$$

The inner integral is

$$\int_0^1 (r - r^2 \cos\theta)\, dr = \left[\frac{1}{2}r^2 - \frac{1}{3}r^3 \cos\theta\right]_0^1 = \frac{1}{2} - \frac{1}{3}\cos\theta \ .$$

Then, the outer integral is

$$\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{3}\cos\theta\right)\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{3}\sin\theta\right]_0^{2\pi} = \pi.$$

Therefore, with *S* as the portion of the *xy*-plane inside the circle of radius 1 centered at the origin, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \pi.$$

Let's try a different surface: Let S be the paraboloid $z = f(x, y) = 1 - x^2 - y^2$ that lies above the xy-plane. Note that C is the same bounding curve. We find $\mathbf{n}$:

$$\mathbf{n} = \langle -f_x, -f_y, 1\rangle = \langle -(-2x), -(-2y), 1\rangle = \langle 2x, 2y, 1\rangle.$$

The curl $\mathbf{F}$ has not changed. Thus,

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \langle -1, -2x, 1 - x\rangle \cdot \langle 2x, 2y, 1\rangle = -3x - 4xy + 1.$$

The region of integration is the same—the interior of the circle of radius 1, centered at the origin. Once again, we use polar coordinates:

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (-3x - 4xy + 1)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (-3r\cos\theta - 4(r\cos\theta)(r\sin\theta) + 1)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (-3r^2\cos\theta - 4r^3\cos\theta\sin\theta + r)\, dr\, d\theta\,.$$

The inside integral, evaluated with respect to r, is

$$\int_0^1 (-3r^2\cos\theta - 4r^3\cos\theta\sin\theta + r)\, dr\, d\theta$$

$$= \left[-r^3\cos\theta - r^4\cos\theta\sin\theta + \frac{1}{2}r^2\right]_0^1$$

$$= -\cos\theta - \cos\theta\sin\theta + \frac{1}{2}\ .$$

Then this is integrated with respect to θ:

$$\int_0^{2\pi}\left(-\cos\theta - \cos\theta\sin\theta + \frac{1}{2}\right) d\theta$$

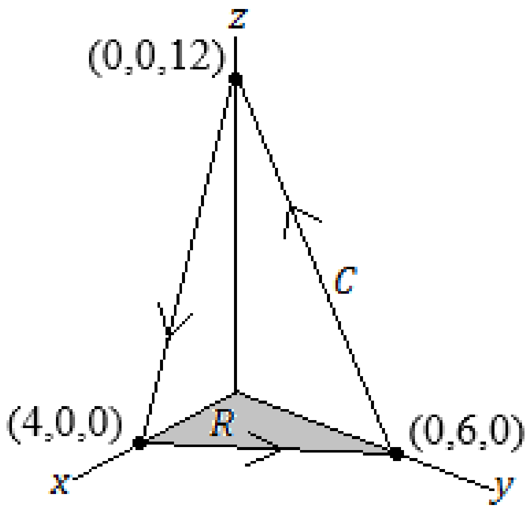
$$= \left[-\sin\theta - \frac{1}{2}\sin^2\theta + \frac{1}{2}\theta\right]_0^{2\pi}$$

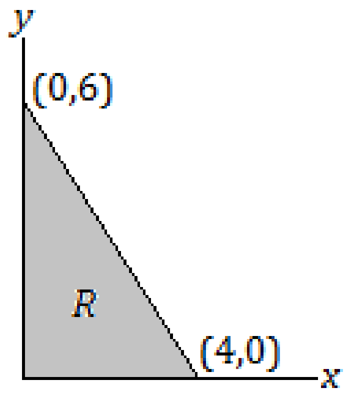
$$= \pi.$$

We got the same result!

Example 2: Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle x + y, zy, 3x \rangle$ and C is the triangle traversed from (4,0,0) to (0,6,0) to (0,0,12), back to (4,0,0). Assume "up" is in the direction of positive z.

Solution: Since no surface is specified, let's use a plane passing through the vertices of the triangle. Below is an image of the path C and the eventual region of integration R:





The plane is $\frac{x}{4} + \frac{y}{6} + \frac{z}{12} = 1$, or $3x + 2y + z = 12$ when fractions are cleared.

We can read off a normal vector from the plane's equation: $\mathbf{n} = \langle 3,2,1 \rangle$. This is a useful vector since it has a 1 in the z position, agreeing with the upward direction.

The curl $\mathbf{F}$ is $\langle -y, -3, -1 \rangle$. Thus,

$$\iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\, dS = \iint_R (-3y - 7)\, dA$$

$$= \int_0^4 \int_0^{6-(3/2)x} (-3y - 7)\, dy\, dx\,.$$

The inside integral is

$$\int_0^{6-(3/2)x} (-3y - 7)\, dy = \left[-\frac{3}{2}y^2 - 7y\right]_0^{6-(3/2)x}$$

$$= -\frac{3}{2}\left(6 - \frac{3}{2}x\right)^2 - 7\left(6 - \frac{3}{2}x\right)$$

$$= -\frac{27}{8}x^2 + \frac{75}{2}x - 96.$$

The outside integral is

$$\int_0^4 \left(-\frac{27}{8}x^2 + \frac{75}{2}x - 96\right) dx = \left[-\frac{9}{8}x^3 + \frac{75}{4}x^2 - 96x\right]_0^4 = -156.$$

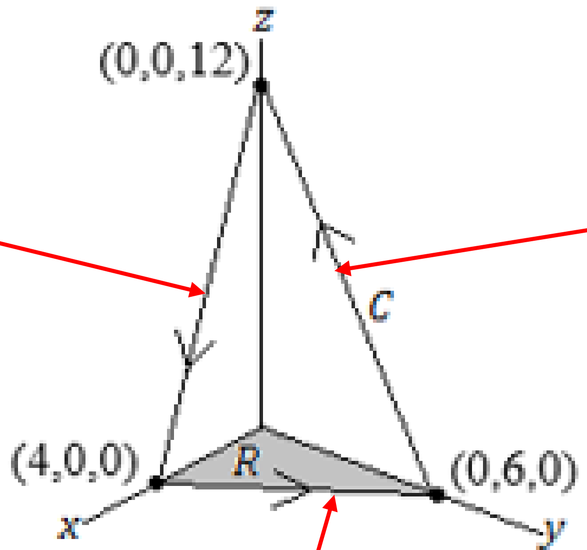
Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r} = -156$. Let's verify this by finding the line integral along each segment of the triangle.

From (0,0,12) to (4,0,0), we have $\mathbf{r}(t) = \langle 4t, 0, 12 - 12t \rangle$ for $0 \leq t \leq 1$. This gives $d\mathbf{r} = \langle 4, 0, -12 \rangle$. Also, $\mathbf{F}(t) = \langle 4t, 0, 12t \rangle$. Therefore,

$$\mathbf{F} \cdot d\mathbf{r} = 4(4t) - 12(12t)$$
$$= 16t - 144t = -128t.$$

Finally, the line integral is

$$\int_0^1 -128t\, dt = -128 \left[\frac{1}{2} t^2\right]_0^1 = -64.$$



From (0,6,0) to (0,0,12), we have $\mathbf{r}(t) = \langle 0, 6 - 6t, 12t \rangle$ for $0 \leq t \leq 1$, so that $d\mathbf{r} = \langle 0, -6, 12 \rangle$. Meanwhile, $\mathbf{F}(t) = \langle 6 - 6t, 72t - 72t^2, 0 \rangle$.

Thus, $\mathbf{F} \cdot d\mathbf{r} = -6(72t - 72t^2) = -432(t - t^2)$, and the line integral is

$$\int_0^1 -432(t - t^2)\, dt$$
$$= -432 \left[\frac{1}{2} t^2 - \frac{1}{3} t^3\right]_0^1 = -72$$

From (4,0,0) to (0,6,0), we have $\mathbf{r}(t) = \langle 4 - 4t, 6t, 0 \rangle$ for $0 \leq t \leq 1$, so that $d\mathbf{r} = \langle -4, 6, 0 \rangle$. Meanwhile,

$$\mathbf{F}(t) = \langle x + y, zy, 3x \rangle$$

$$= \langle (4 - 4t) + (6t), (0)(6t), 3(4 - 4t) \rangle \quad \begin{cases} x = 4 - 4t \\ y = 6t \\ z = 0 \end{cases}$$

or after simplification, $\mathbf{F}(t) = \langle 4 + 2t, 0, 12 - 12t \rangle$. The dot product is $\mathbf{F} \cdot d\mathbf{r} = -4(4 + 2t) = -16 - 8t$, and the line integral is

$$\int_0^1 (-16 - 8t)\, dt = [-16t - 4t^2]_0^1 = -20.$$

The sum of these three line integrals is

$$-20 - 72 - 64 = -156,$$

agreeing with the result found by Stokes Theorem.