Surface Integrals

Scott Surgent

Let $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ parametrically describe a surface S in R^3. Then the **surface area** of S over a region of integration R is given by

$$\iint_S dS = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, dA\,.$$

If the surface is defined explicitly in the form $z = f(x,y)$, then the surface can be parametrized as

$$\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle.$$

Its partial derivatives are

$$\mathbf{r}_x = \langle 1,0, f_x(x,y)\rangle \quad \text{and} \quad \mathbf{r}_y = \langle 0,1, f_y(x,y)\rangle.$$

The cross product is

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x(x,y), -f_y(x,y), 1\rangle,$$

and the magnitude of this cross product is

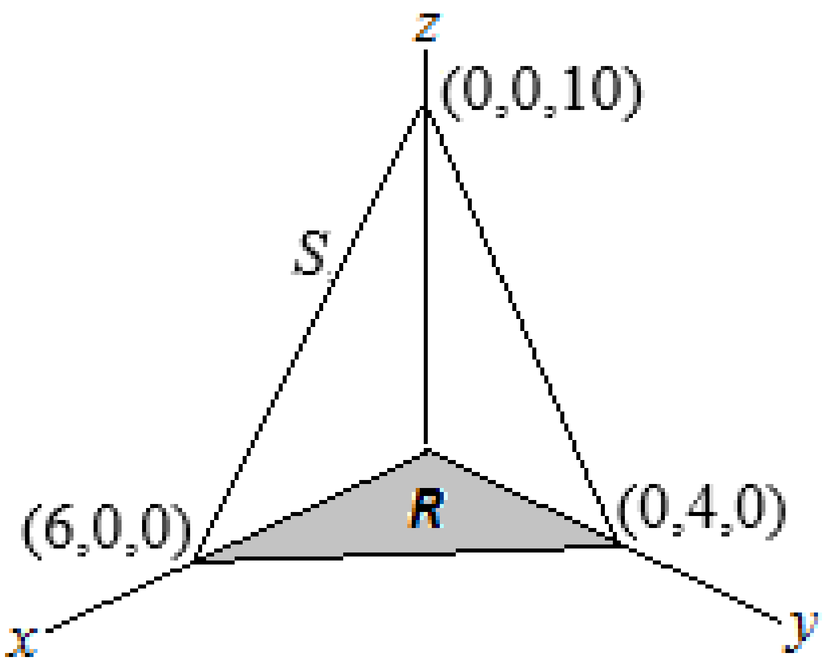
$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(-f_x(x,y)\right)^2 + \left(-f_y(x,y)\right)^2 + 1^2} = \sqrt{\left(f_x(x,y)\right)^2 + \left(f_y(x,y)\right)^2 + 1}.$$

Thus, in the case of a surface being described by an explicitly-defined function, the surface area of the surface S over a region of integration R is

$$\iint_S dS = \iint_R \sqrt{\left(f_x(x,y)\right)^2 + \left(f_y(x,y)\right)^2 + 1}\, dx\, dy\,.$$

Example 1: Find the surface area of the plane with intercepts (6,0,0), (0,4,0) and (0,0,10) that is in the first octant.

Solution: The plane's equation is $\frac{x}{6}+\frac{y}{4}+\frac{z}{10}=1$, or $10x+15y+6z=60$. Below is a sketch of the surface S, the plane in the first octant, and its region of integration R in the xy-plane:



Solving for z, we have $z=10-\frac{5}{3}x-\frac{5}{2}y$. Therefore, the plane can be written parametrically:

$$\mathbf{r}(x,y)=\left\langle x,y,10-\frac{5}{3}x-\frac{5}{2}y\right\rangle.$$

Its partial derivatives are $\mathbf{r}_x=\left\langle 1,0,-\frac{5}{3}\right\rangle$ and $\mathbf{r}_y=\left\langle 0,1,-\frac{5}{2}\right\rangle$, and the cross product is

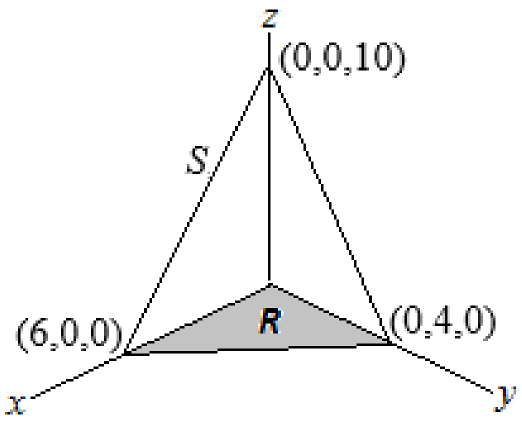
$$\mathbf{r}_x\times\mathbf{r}_y=\left\langle \frac{5}{3},\frac{5}{2},1\right\rangle.$$

Therefore, the magnitude is

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{5}{3}\right)^2 + \left(\frac{5}{2}\right)^2 + 1^2} = \sqrt{\frac{361}{36}} = \frac{19}{6}\,.$$

The surface area is

$$\iint_S dS = \iint_R |\mathbf{r}_x \times \mathbf{r}_y|\, dA = \frac{19}{6}\iint_R dA\,.$$



Note that $\iint_R dA$ is the area of the region of integration R.

Using the formula for area of a triangle, the area of R is $\frac{1}{2}(6)(4) = 12$.

Thus, the surface area of the plane $z = 10 - \frac{5}{3}x - \frac{5}{2}y$ in the first octant is $\frac{19}{6}(12) = 38$ square units.

Example 2: Find the surface area of the paraboloid $z = 9 - x^2 - y^2$ that extends above the *xy*-plane.

Solution: Parametrically, the paraboloid is $\mathbf{r}(x, y) = \langle x, y, 9 - x^2 - y^2 \rangle$, and its partial derivatives are $\mathbf{r}_x = \langle 1,0, -2x \rangle$ and $\mathbf{r}_y = \langle 0,1, -2y \rangle$. Therefore, their cross product is

$$\mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1 \rangle,$$

and the magnitude of the cross product is

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{(2x)^2 + (2y)^2 + 1^2} = \sqrt{4x^2 + 4y^2 + 1}.$$

The paraboloid intersects the *xy*-plane ($z = 0$) at a circle of radius 3, centered at the origin, so that the region of integration R is given by $x^2 + y^2 \leq 9$. Therefore, the surface area of the paraboloid $z = 9 - x^2 - y^2$ that extends above the *xy*-plane is given by

$$\iint_S dS = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA.$$

In rectangular coordinates, this is a difficult integrand to integrate. Instead, we use polar coordinates to rewrite this surface-area integral in terms of r and θ:

$$\iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1}\, r\, dr\, d\theta\, .$$

The inside integral is evaluated first:

$$\int_0^3 \sqrt{4r^2 + 1}\, r\, dr = \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_0^3 = \frac{1}{12}\left(37^{3/2} - 1\right).$$

Then, the outside integral is evaluated to find the surface area:

$$\frac{1}{12}\left(37^{3/2} - 1\right) \int_0^{2\pi} d\theta = \frac{\pi}{6}\left(37^{3/2} - 1\right), \text{or about } 117.32 \text{ units}^2.$$

General Surface Integrals

The area of a surface S in R^3 defined parametrically by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a region of integration R in the input-variable (uv) plane is given by

$$\iint_S dS = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA\,.$$

Let $w = f(x,y,z)$ be a function defined over this surface. The **surface integral**, where $f(\mathbf{r}(u,v)) = f(x(u,v), y(u,v), z(u,v))$, is given by

$$\iint_S f(\mathbf{r}(u,v))\, dS = \iint_R f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA\,.$$

When the surface S is defined explicitly by a function $z = g(x, y)$, then $\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle$, and the surface integral can be rewritten

$$\iint_S f(x, y, z)\, dS = \iint_R f\big(x, y, g(x, y)\big)\sqrt{\big(g_x(x, y)\big)^2 + \big(g_y(x, y)\big)^2 + 1}\, dA,$$

Where $dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \sqrt{\big(g_x(x, y)\big)^2 + \big(g_y(x, y)\big)^2 + 1}\, dA.$

Surface area integrals are a special case of surface integrals, where $f(x, y, z) = 1$.

Example 3: Let S be the surface $z = 12 - 4x - 3y$ contained in the first quadrant. Find $\iint_S (x + yz)\, dS$.

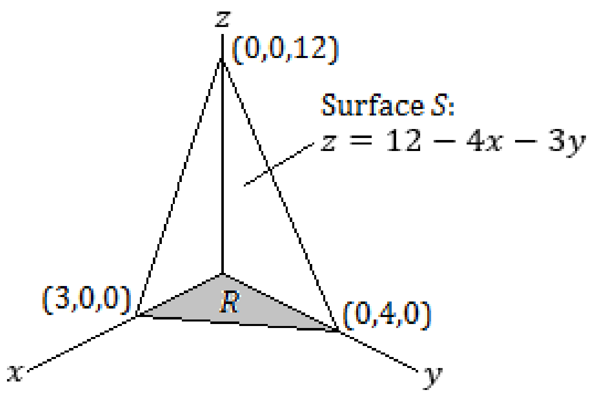
Solution: Here, $z = g(x, y) = 12 - 4x - 3y$, so that $g_x = -4$ and $g_y = -3$. Thus, dS is

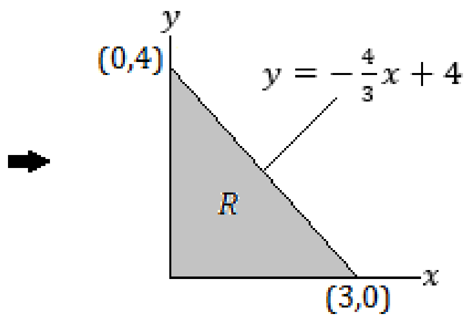
$$dS = \sqrt{(-4)^2 + (-3)^2 + 1} = \sqrt{26}\, dA.$$

The integrand is written in terms of x and y, with $z = 12 - 4x - 3y$:

$$x + yz = x + y(12 - 4x - 3y) = x + 12y - 4xy - 3y^2$$

The region of integration R is the footprint of the surface S projected onto the xy-plane. Below is a sketch of S and its region of integration R. Letting $dA = dy\, dx$, we have $0 \le y \le -\frac{4}{3}x + 4$ and $0 \le x \le 3$ as the bounds of R:





The surface integral is now

$$\iint_S (x + yz)\, dS = \iint_R (x + 12y - 4xy - 3y^2)\sqrt{26}\; dA$$

$$= \sqrt{26}\int_0^3 \int_0^{-(4/3)x+4} (x + 12y - 4xy - 3y^2)\; dy\, dx.$$

The inside integral is

$$\int_0^{-(4/3)x+4} (x + 12y - 4xy - 3y^2)\; dy = [xy + (6 - 2x)y^2 - y^3]_0^{-(4/3)x+4}$$

Note that the two middle terms, $12y - 4xy$, can be written $(12 - 4x)y$, which gives $(6 - 2x)y^2$ after integration with respect to y. Substituting and simplifying, we obtain

$$x\left(-\frac{4}{3}x + 4\right) + (6 - 2x)\left(-\frac{4}{3}x + 4\right)^2 - \left(-\frac{4}{3}x + 4\right)^3 = -\frac{32}{27}x^3 + \frac{28}{3}x^2 - 28x + 32.$$

This is now integrated with respect to *x*:

$$\sqrt{26}\int_0^3\left(-\frac{32}{27}x^3+\frac{28}{3}x^2-28x+32\right)dx$$

$$=\sqrt{26}\left[-\frac{8}{27}x^4+\frac{28}{9}x^3-14x^2+32x\right]_0^3$$

$$=30\sqrt{26}.$$