MAT267: Vectors

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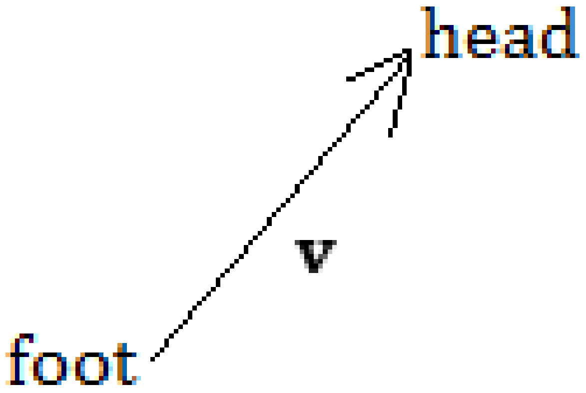
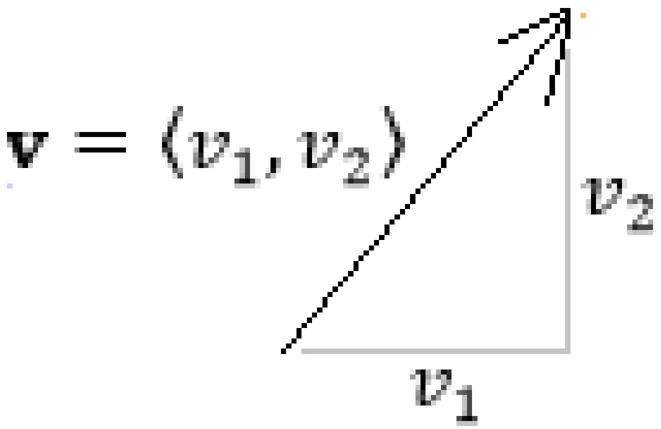
For purposes of applications in calculus and physics, a **vector** has both a direction and a magnitude (length) and is usually represented by an arrow. The start of the arrow is the vector's *foot*, and the end is its *head*. A vector is usually labelled in boldface, such as **v**.

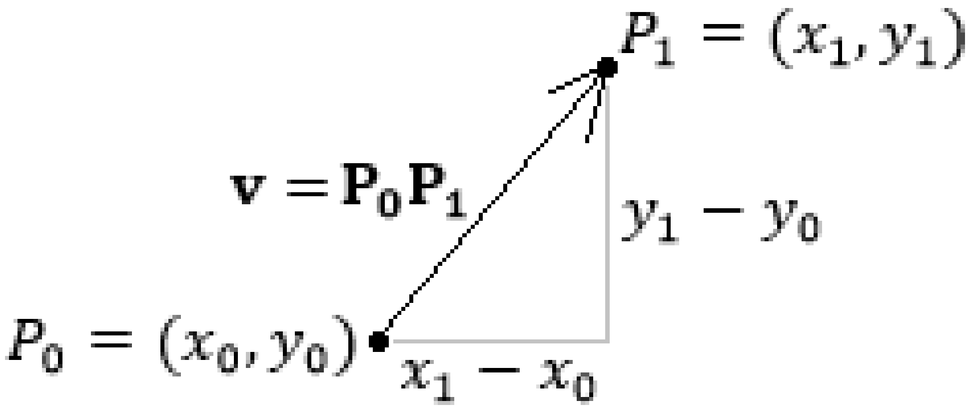


In an *xy*-axis system (R^2), a vector is written $\mathbf{v} = \langle v_1, v_2 \rangle$, which means that from the foot of **v**, move v_1 units in the *x* direction, and v_2 units in the *y* direction, to arrive at the vector's head. The values v_1 and v_2 are the vector's **components**. In R^3, a vector has three components and is written $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.



Given two points, $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$ in R^2, a vector $\mathbf{P_0P_1}$ can be drawn with its foot at P_0 and head at P_1, where $\mathbf{P_0P_1} = \langle x_1 - x_0, y_1 - y_0 \rangle$. In R^3, the vector is expressed

$$\mathbf{P_0P_1} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle.$$



A **scalar** is a number only, with no implied direction. Scalars are chosen from the set of real numbers *R*.

The **magnitude** of a vector **v** is found by the Pythagorean Formula:

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2} \quad (\text{in } R^2)$$

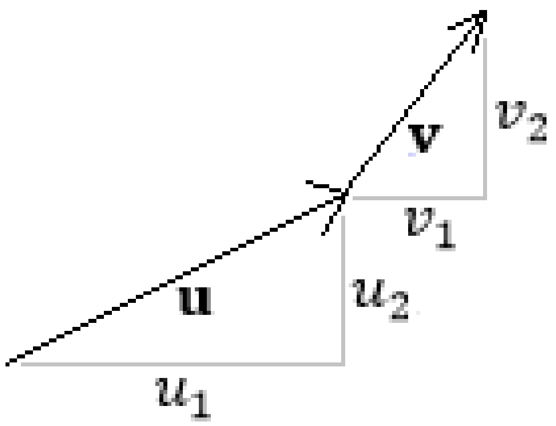
or $$\sqrt{v_1^2 + v_2^2 + v_3^2} \quad (\text{in } R^3).$$

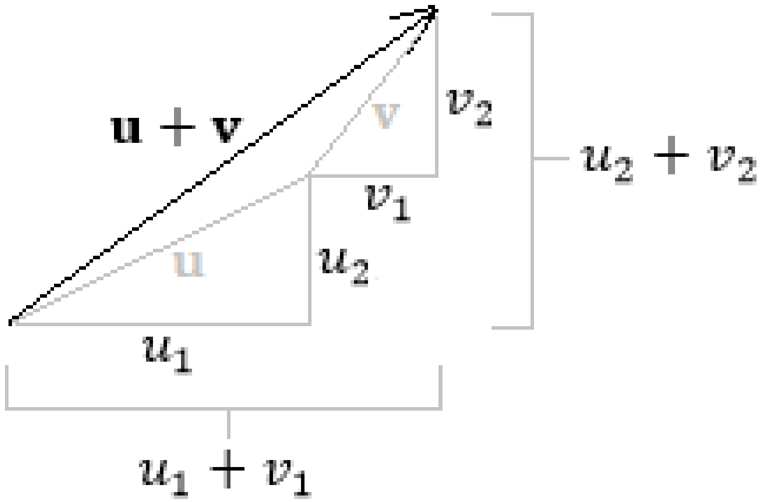
The notation $|\mathbf{v}|$ is the magnitude of **v** and is always a non-negative value. The expression $|\mathbf{v}|$ is a scalar.

To add two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, add the respective components:

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle.$$

Place the foot of **v** at the head of **u**, then sketch a vector that begins at the foot of **u** and ends at the head of **v**. The vector $\mathbf{u} + \mathbf{v}$ is called the **resultant**.





Pay attention to notation. Parentheses ( ) are used to represent points, and angled brackets ⟨ ⟩ are used to represent vectors.

A vector may be multiplied by any real number c, called a **scalar multiple**.

For example, if $\mathbf{u}$ is a vector, then $2\mathbf{u} = \mathbf{u} + \mathbf{u} = \langle 2u_1, 2u_2 \rangle$, which results in a vector $2\mathbf{u}$ that is twice the magnitude of $\mathbf{u}$.

Scalars act as coefficients when multiplied to a vector.

In general, for a vector $\mathbf{v}$ and a scalar c, the magnitude of $c\mathbf{v}$ is $|c\mathbf{v}| = |c||\mathbf{v}|$, where $|c|$ is the absolute value of c.

Two non-zero vectors $\mathbf{u}$ and $\mathbf{v}$ are **parallel** if one can be written as a scalar multiple of the other, $\mathbf{u} = c\mathbf{v}$ for some non-zero scalar c.

There are two **closure properties** of vectors:

C1. If **u** and **v** are two vectors in R^2 (or R^3), then their vector sum **u** + **v** is also in R^2 (or R^3).

C2. If **u** is a vector in R^2 (or R^3), then for any scalar c, its scalar multiple c**u** is also in R^2 (or R^3).

The **structural properties** of vectors are:

P1. *Commutativity*: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$. Vectors can be added in any order.

P2. *Associativity*: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

P3. *Additive Identity*: $\mathbf{0} = \langle 0,0 \rangle$ or $\langle 0,0,0 \rangle$, with the property that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$. Thus, $\mathbf{0}$ is called the **zero vector**, and is a single point, with a magnitude of 0: $|\mathbf{0}| = 0$, and if $|\mathbf{v}| = 0$, then $\mathbf{v} = \mathbf{0}$.

P4. *Additive Inverse*: For any non-zero vector $\mathbf{u}$, the vector $-\mathbf{u}$ exists with the property that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. Visually, $-\mathbf{u}$ has the same magnitude as $\mathbf{u}$, but points in the opposite direction. Subtraction of two vectors is now defined: $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \mathbf{u} + (-1\mathbf{v})$.

Structural Properties, continued:

P5. *Distributivity of a scalar across vectors*: If c is a scalar, then $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.

P6. *Distributivity of a vector across scalars*: If c and d are scalars, then $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.

P7. *Associativity and Commutativity of Scalars*: $cd\mathbf{u} = c(d\mathbf{u}) = (cd)\mathbf{u} = d(c\mathbf{u}) = dc\mathbf{u}$.

P8. *Multiplicative scalar identity*: $1\mathbf{v} = \mathbf{v}$.

Any set V for which the two closure properties and the eight structural properties are true for all elements in V and for all real-number scalars is called a **vector space**.

Elements of a vector space are called vectors.

Common vector spaces are R^n, where *n* is any non-negative integer.

The *xy*-axis system R^2 is a vector space, where any ordered pair (*a*,*b*) can be thought of as a vector from (0,0) to (*a*,*b*). In this manner, the elements of R^2 are vectors of the form $\langle a, b\rangle$, and all of the closure and structural properties listed above are met.

Similarly, for R^3, the real line R^1 $(= R)$, and the trivial space $R^0 = \mathbf{0}$ are vector spaces.

Given any non-zero vector **v**, the **unit vector** of **v** is found by multiplying **v** by $\frac{\mathbf{1}}{|\mathbf{v}|}$.

The unit vector has magnitude 1. That is, $\left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = \frac{\mathbf{1}}{|\mathbf{v}|}|\mathbf{v}| = 1$.

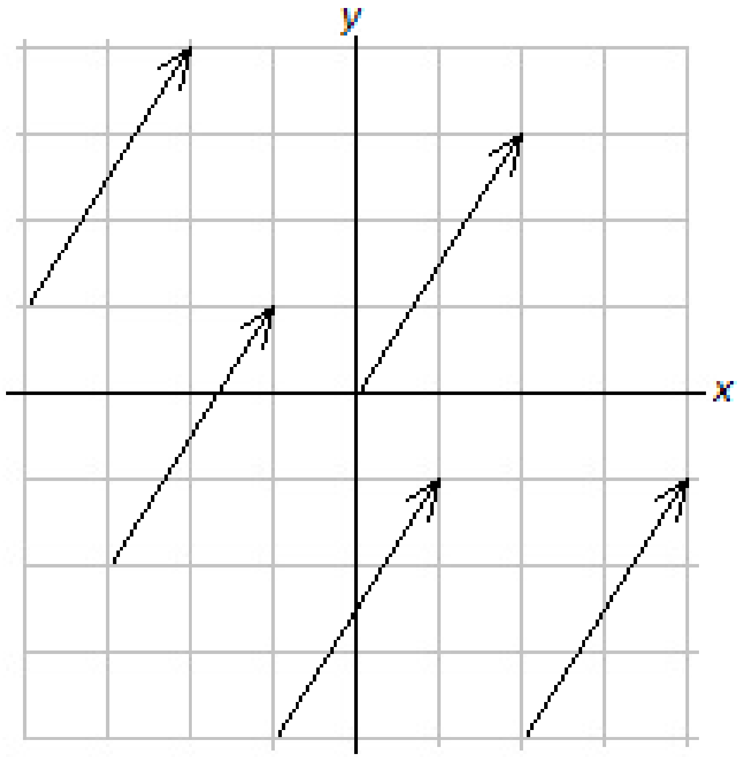
The unit vector of **v** is any vector of length 1 parallel and in the same direction to **v**. Common notation for the unit vector of **v** is $\hat{\mathbf{v}}$ ("**v**-hat") or $\mathbf{v}_{unit}$.

In R^2, the vectors $\mathbf{i} = \langle 1,0 \rangle$ and $\mathbf{j} = \langle 0,1 \rangle$ are called the **standard orthonormal basis vectors**, which allows us to write a vector $\mathbf{v} = \langle v_1, v_2 \rangle = v_1\mathbf{i} + v_2\mathbf{j}$.

In R^3, the standard orthonormal basis vectors are $\mathbf{i} = \langle 1,0,0 \rangle$, $\mathbf{j} = \langle 0,1,0 \rangle$ and $\mathbf{k} = \langle 0,0,1 \rangle$. The notation $\langle v_1, v_2 \rangle$ and $v_1\mathbf{i} + v_2\mathbf{j}$ to represent a vector in R^2 can be used interchangeably.

Example 1: Sketch $\mathbf{u} = \langle 2,3 \rangle = 2\mathbf{i} + 3\mathbf{j}$.

Solution: From any starting point, move 2 units in the *x* (horizontal) direction, and 3 units in the *y* (vertical) direction. Below are five copies of the vector **u**.



The foot can be placed anywhere. Multiple copies of the same vector can be drawn using different starting points. The position of a vector relative to a coordinate axis system is not relevant. As long as its direction and magnitude are not changed, it is considered to be the same vector.

Example 2: Given $\mathbf{v} = \langle 4, -5 \rangle$, find $|\mathbf{v}|$, and the unit vector of **v**.

Solution: The magnitude of **v** is

$$|\mathbf{v}| = \sqrt{(4)^2 + (-5)^2} = \sqrt{16 + 25} = \sqrt{41}.$$

The unit vector of **v** is

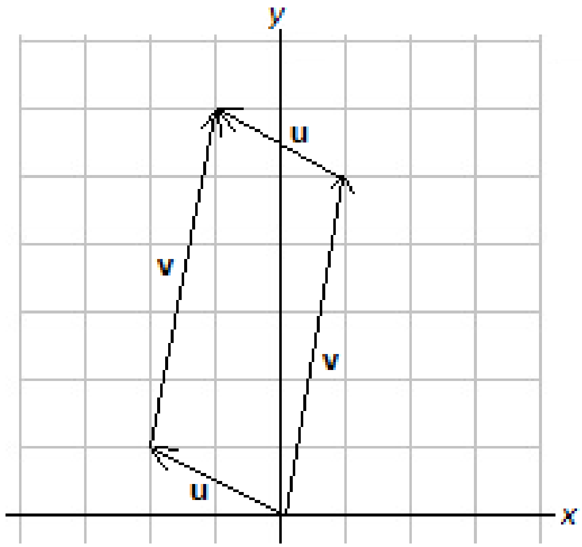
$$\hat{\mathbf{v}} = \mathbf{v}_{unit} = \frac{1}{\sqrt{41}} \langle 4, -5 \rangle = \left\langle \frac{4}{\sqrt{41}}, -\frac{5}{\sqrt{41}} \right\rangle.$$

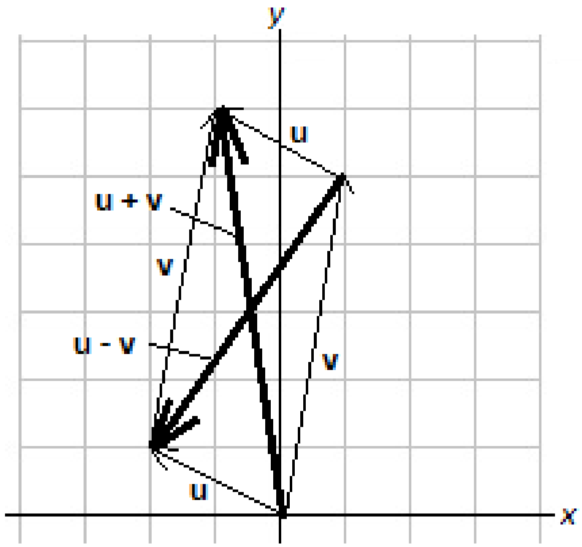
Example 3. Given $\mathbf{u} = \langle -2,1 \rangle$ and $\mathbf{v} = \langle 1,5 \rangle$, Find and sketch (a) **u** + **v**, and (b) **u** – **v**.

Solution: We have

$$\mathbf{u} + \mathbf{v} = \langle -2 + 1, 1 + 5 \rangle = \langle -1, 6 \rangle$$

$$\mathbf{u} - \mathbf{v} = \langle -2 - 1, 1 - 5 \rangle = \langle -3, -4 \rangle.$$





Example 4: Given the vectors $\mathbf{u} = \langle 1, -4, 2 \rangle$ and $\mathbf{v} = \langle -5, 8, 3 \rangle$, find $\mathbf{w}$ where $3\mathbf{w} + 2\mathbf{u} = -\mathbf{v}$.

Solution: Using algebra, solve for **w**:

$$3\mathbf{w} + 2\mathbf{u} = -\mathbf{v}$$

$$3\mathbf{w} = -\mathbf{v} - 2\mathbf{u}$$

$$\mathbf{w} = -\frac{1}{3}\mathbf{v} - \frac{2}{3}\mathbf{u}.$$

Now, substitute **u** and **v**, and simplify:

$$\mathbf{w} = -\frac{1}{3}\mathbf{v} - \frac{2}{3}\mathbf{u}$$

$$= -\frac{1}{3}\langle -5, 8, 3 \rangle - \frac{2}{3}\langle 1, -4, 2 \rangle$$

$$= \left\langle \frac{5}{3}, -\frac{8}{3}, -1 \right\rangle + \left\langle -\frac{2}{3}, \frac{8}{3}, -\frac{4}{3} \right\rangle$$

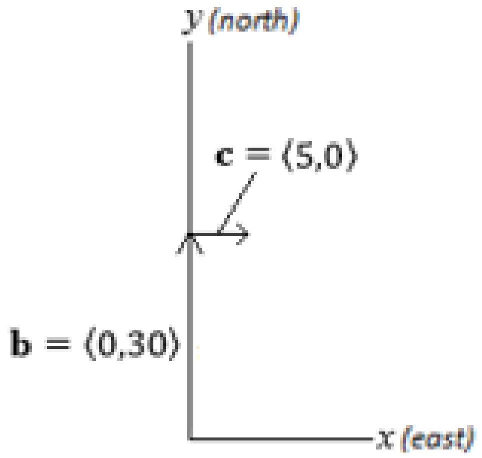
$$= \left\langle 1, 0, -\frac{7}{3} \right\rangle.$$

Example 5: Given $\mathbf{v} = \langle -1,5,2 \rangle$, find a vector $\mathbf{w}$ in the same direction as $\mathbf{v}$, with magnitude 3.

Solution: The unit vector is $\mathbf{v}_{unit} = \frac{1}{\sqrt{30}}\langle -1,5,2 \rangle$. Then multiply by 3:

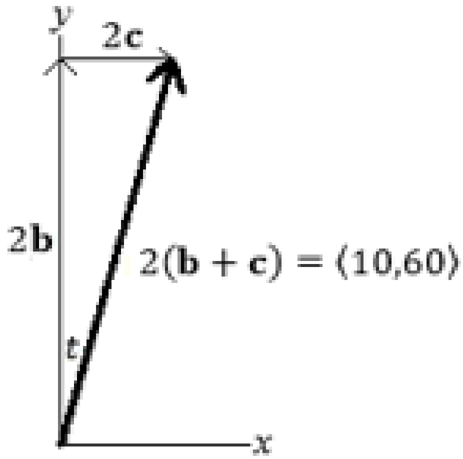
$$\mathbf{w} = 3\mathbf{v}_{unit} = \frac{3}{\sqrt{30}}\langle -1,5,2 \rangle.$$

Example 6: A boat travels north at 30 miles per hour. Meanwhile, the current is moving toward the east at 5 miles per hour. If the boat's captain does not account for the current, the boat will drift to the east of its intended destination. After two hours, find (a) the boat's position as a vector, (b) the distance the boat travelled, and (c) the boat's position as a bearing.



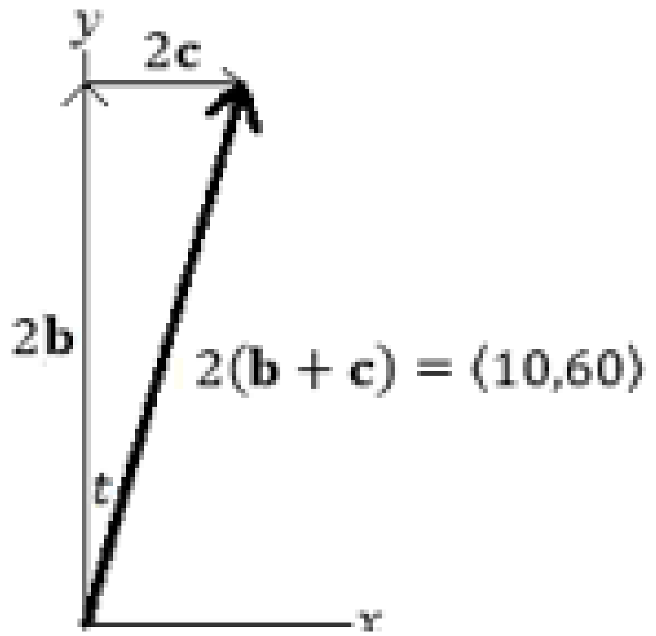
Solution. Superimpose an *xy*-axis system, so that the positive *y*-axis North, and the positive *x*-axis is East. Thus, the boat's vector can be represented by $\mathbf{b} = \langle 0,30 \rangle$, and the current's vector by $\mathbf{c} = \langle 5,0 \rangle$.

a) The boat's position after two hours will be $2(\mathbf{b} + \mathbf{c}) = 2\langle 5,30 \rangle = \langle 10,60 \rangle$. From the boat's starting point, the boat moved 10 miles east and 60 miles north.

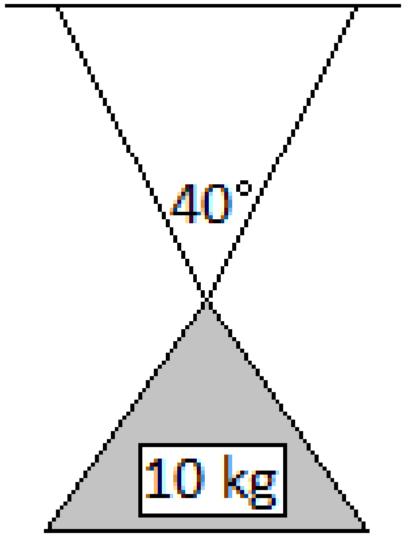


b) The boat travelled a distance of $|2(\mathbf{b}+\mathbf{c})| = 2\sqrt{5^2+30^2} = 2\sqrt{925}$, or about 60.83 miles.

c) Viewing a drawing below, we see that we can find the angle t using inverse trigonometry. Thus, $t = \tan^{-1}(10/60) \approx$ 9.46 degrees East of North.



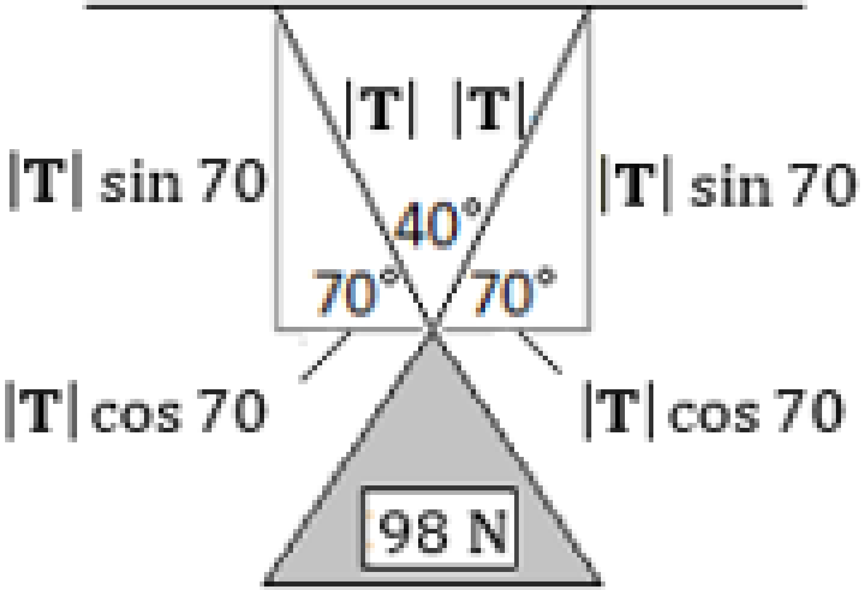
Example 7: A 10 kg mass hangs by two symmetric cables from a ceiling such that the cables meet at a 40-degree angle at the mass itself. Find the tension (in Newtons) on each cable.

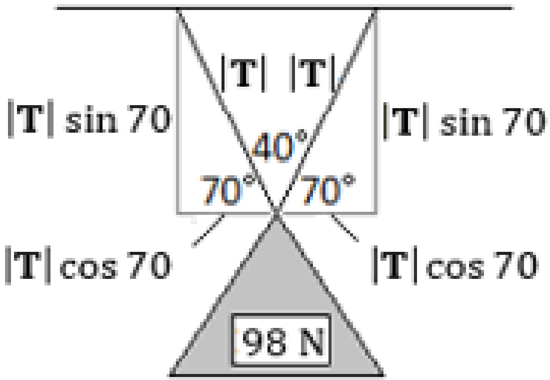


Solution: The force of the mass with respect to gravity is

$$F = ma = (10\text{ kg})\left(9.8\,\frac{\text{m}}{\text{s}^2}\right) = 98\text{ N}.$$

Let **|T|** be the tension on one cable. We decompose **|T|** into its vertical and horizontal components:





The two horizontal components sum to 0, since the forces cancel one another, while the two vertical components support the 98 N downward force. Thus, we have

$$2|\mathbf{T}|\sin 70 = 98.$$

Solving for $|\mathbf{T}|$ we, we obtain $|\mathbf{T}| = \frac{98}{2\sin 70} = 52.14$ N.

Each cable has a tension of about 52.14 N. If the angle at which the cables meet was larger, the tensions would be greater. For example, if the cables were to meet at the mass at an angle of 150 degrees, then each cable would have a tension of

$$|\mathbf{T}| = \frac{98}{2\sin 15} = 189.32 \text{ N}$$