MAT267 – The xyz Coordinate Axis System

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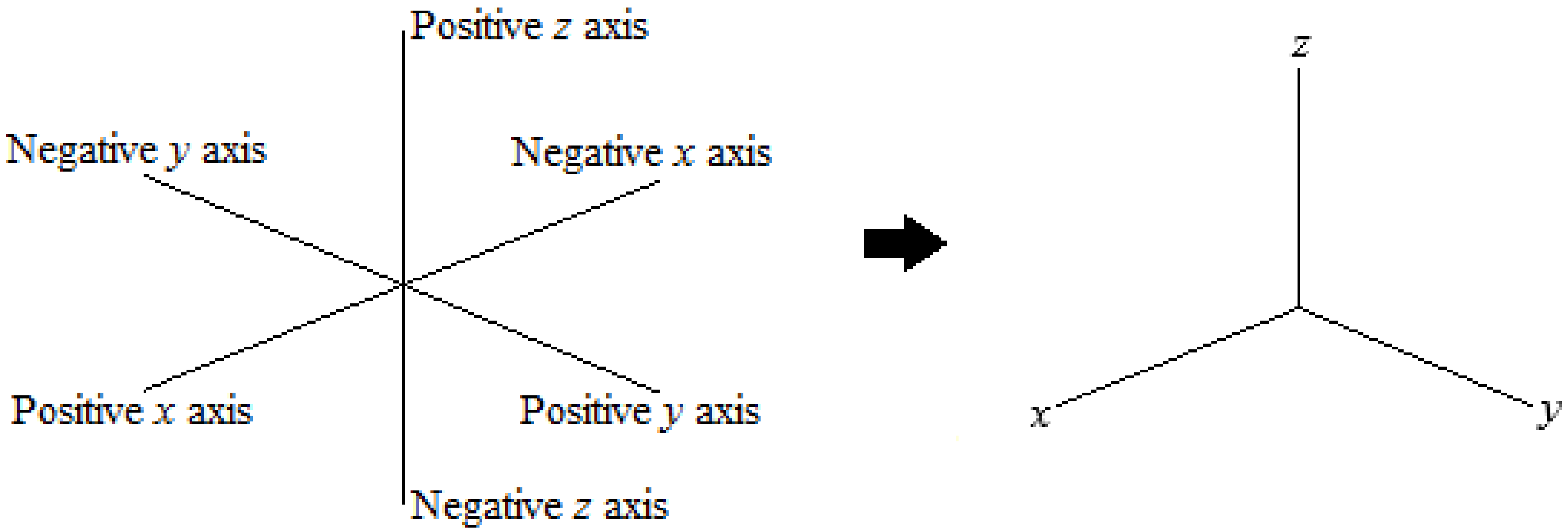
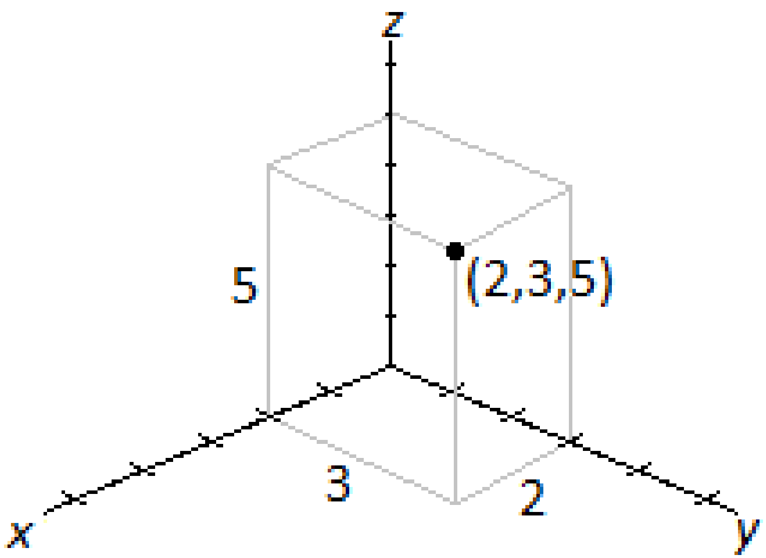
The ***xyz* coordinate axis system**, denoted R^3, is represented by three real number lines meeting at a common point, called the origin. The three number lines are called the ***x*-axis**, the ***y*-axis**, and the ***z*-axis**. Together, the three axes are called the **coordinate axes**.

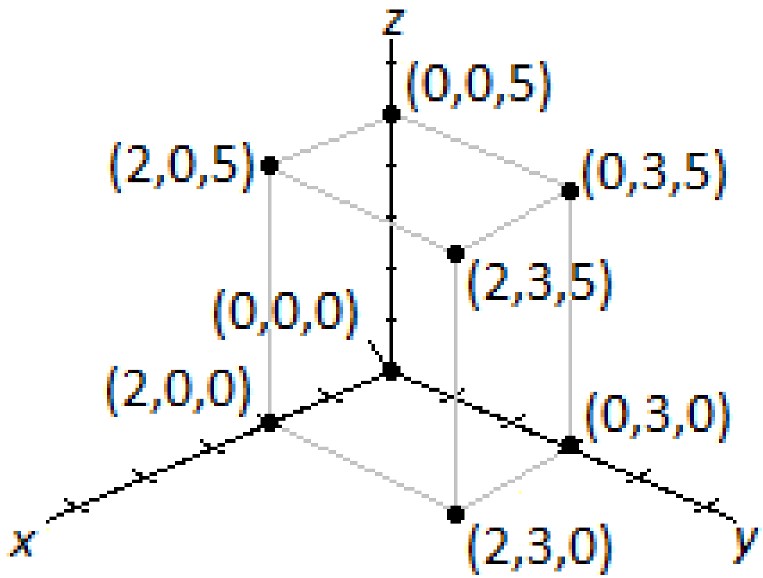


The three axes divide R^3 into eight regions, called **octants**. The region in which *x*, *y* and *z* are positive is called the **first octant** or the **positive octant**.

A point is represented by an **ordered triple** (x, y, z), in which from the origin (whose ordered triple is (0,0,0)), one moves *x* units along the *x*-axis, then *y* units parallel to the *y*-axis, and then *z* units parallel to the *z*-axis, to arrive at the point. The values *x*, *y* and *z* are the **coordinates** of the point

Example 1: Represent the point (2,3,5) on an *xyz*-coordinate axis system.

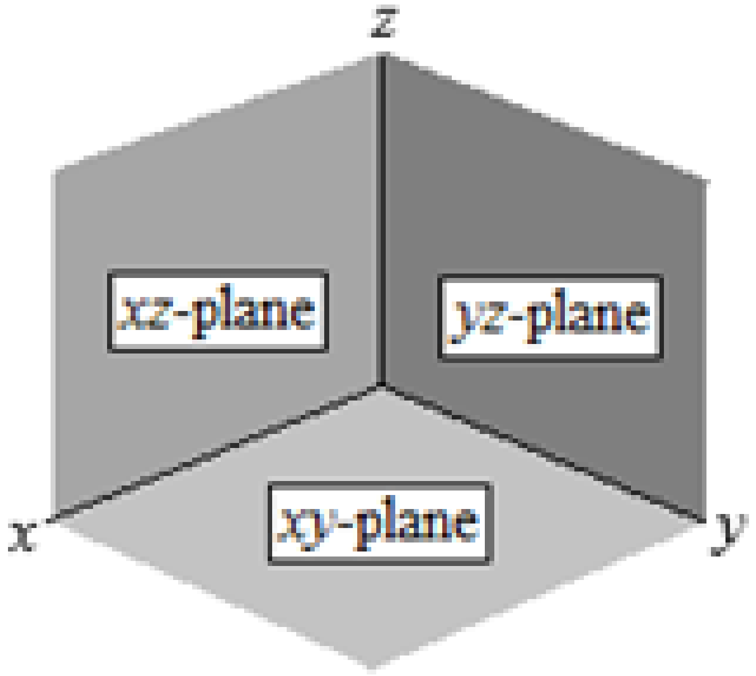




The point (2,3,0) is called a **projection** of (2,3,5) onto the *xy*-plane, found by setting $z = 0$. Other projections can be found similarly.

The three coordinate axes, taken two at a time, form three **coordinate planes**.

- The x-axis and the y-axis form the ***xy*-coordinate plane** and contains points whose ordered triples are of the form $(x, y, 0)$. The equation $z = 0$ represents the xy-plane.
- The x-axis and the z-axis form the ***xz*-coordinate plane** and contains points whose ordered triples are of the form $(x, 0, z)$. The equation $y = 0$ represents the xz-plane.
- The y-axis and the z-axis form the ***yz*-coordinate plane** and contains points whose ordered triples are of the form $(0, y, z)$. The equation $x = 0$ represents the yz-plane.



Example 2: The point (100,6,4) is closest to which coordinate plane?

Solution: Since the *z*-value of 4 is the smallest of the three coordinates, the point (100,6,4) is closest to the *xy* coordinate plane.

Example 3: Given the point $(4,-1,2)$, find its projections onto the *xy*-plane, the *xz*-plane and the *yz*-plane.

Solution: The *xy*-plane is described by the equation *z* = 0, so the projection of $(4,-1,2)$ onto the *xy*-plane is $(4,-1,0)$. Similarly, the projection of $(4,-1,2)$ onto the *xz*-plane is $(4,0,2)$, and $(4,-1,2)$ onto the *yz*-plane is $(0,-1,2)$.

Example 4: Given the point $(4, -1, 2)$, find its reflections across the *xy*-plane, the *xz*-plane, the *yz*-plane, and the origin.

Solution: Points reflected across the *xy*-plane are found by negating the *z* coordinate. Thus, the reflection of $(4, -1, 2)$ across the *xy*-plane is $(4, -1, -2)$.

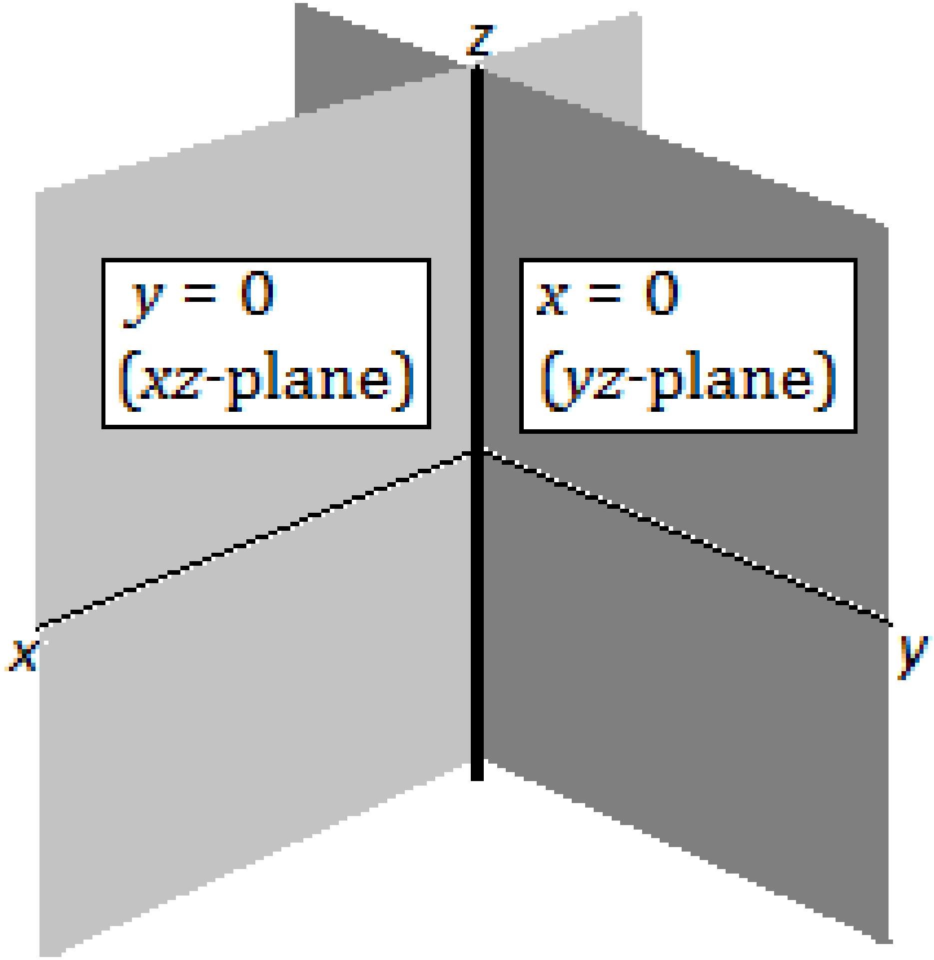
In a similar way, the reflection of $(4, -1, 2)$ across the *xz*-plane is $(4, 1, 2)$, and the reflection of $(4, -1, 2)$ across the *yz*-plane is $(-4, -1, 2)$.

To reflect across the origin, we negate all three coordinates. This is equivalent to reflecting a point across the *xy*-plane, then the *xz*-plane, then the *yz*-plane (in any order). Thus, the reflection of $(4, -1, 2)$ across the origin is $(-4, 1, -2)$.

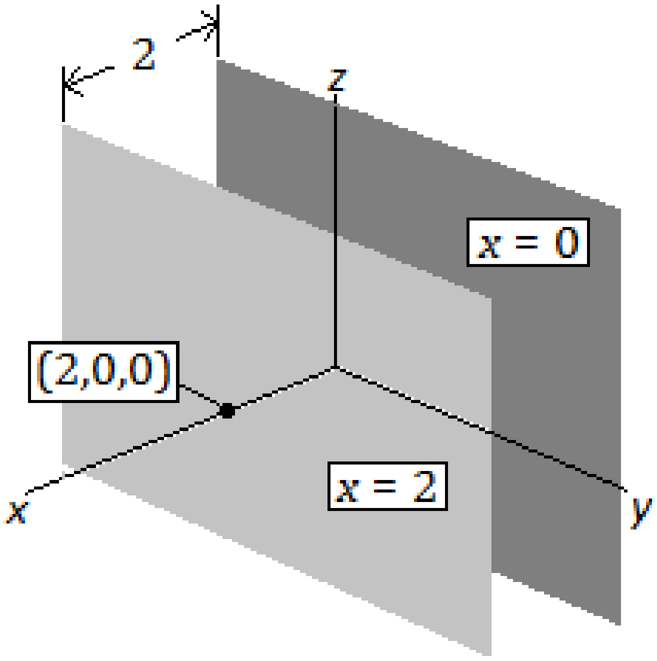
Example 5 Describe the intersection of the planes $x = 0$ and $y = 0$.

Solution: The equation $x = 0$ is the yz-plane, and the equation $y = 0$ is the xz-plane, and they intersect at the z-axis. Points on the z-axis are described using set notation:

$$\{(x, y, z) \mid x = 0, y = 0, z \in R\}.$$



Example 6: Describe the equation $x = 2$ as it appears in R^3.



Solution: The equation $x = 2$ includes all points of the form $(2, y, z)$. More generally, it can be described using set notation:

$$\{(x, y, z)|x = 2, y \in R, z \in R\}.$$

It is a plane that is parallel to the *yz*-plane shifted two units in the positive *x* direction.

The equation $x = 2$ does not imply any restriction on the variables *y* and *z*. They can assume any real number value.

It is important to remember the "space" in which $x = 2$ is defined. In R^3, it is a plane. In R^2, it would be a vertical line passing through (2,0). In R^1 (or R), it is a point on the real number line.

Distance & Midpoint

Given two points $A = (x_0, y_0, z_0)$ and $B = (x_1, y_1, z_1)$ in R^3, the **distance** between A and B is given by

$$D_{A,B} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2},$$

and the **midpoint** between A and B is given by

$$M_{A,B} = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}, \frac{z_0 + z_1}{2}\right).$$

Note that the distance formula is the Pythagorean formula, and that the midpoint formula simply calculates the arithmetic mean (one at a time) of the *x*-coordinates, the *y*-coordinates and the *z*-coordinates.

Example 7: Given $A = (-2,1,4)$ and $B = (5,0,-7)$. Find the distance between A and B, and the midpoint of A and B.

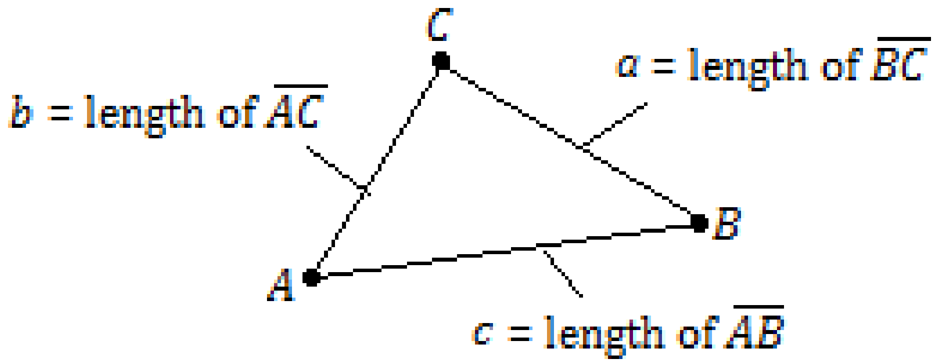
Solution: The distance between A and B is

$$\begin{aligned} D_{A,B} &= \sqrt{\left(5-(-2)\right)^2 + (0-1)^2 + (-7-4)^2} \\ &= \sqrt{7^2 + (-1)^2 + (-11)^2} \\ &= \sqrt{171} \\ &\approx 13.077 \text{ units.} \end{aligned}$$

The midpoint between A and B is

$$M_{A,B} = \left(\frac{-2+5}{2}, \frac{1+0}{2}, \frac{4+(-7)}{2}\right) = \left(\frac{3}{2}, \frac{1}{2}, -\frac{3}{2}\right).$$

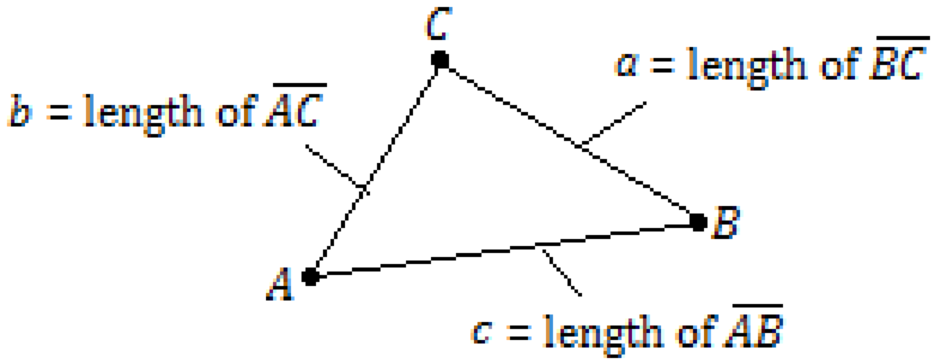
Triangles & Collinearity



Three points A, B and C form a **triangle** in that A, B and C are the vertices (corners) of the triangle, and that line segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ form the sides (edges).

Letting a, b and c represent the lengths of the sides of a triangle, and assuming c is the largest of the three values, the **triangle inequality** states that $c \leq a + b$, which states that the longest side of a triangle cannot be greater than the sum of the lengths of the two shorter sides.

If $c = a + b$, then the length of the longest side is exactly the sum of the lengths of the two shorter sides, which can only happen when points A, B and C lie on a common line. In such a case, points A, B and C are **collinear**.



The three side-lengths of a triangle are related by the **law of cosines**:

$$c^2 = a^2 + b^2 - 2ab\cos\theta\,,$$

where c is assumed to be the length of the longest side and θ is the angle formed at point C, where side segments $\overline{AC}$ and $\overline{BC}$ meet. If $\theta = 90°$, then $\cos\theta = 0$, and we have the Pythagorean Formula, which relates the three side-lengths of a **right triangle**:

$$c^2 = a^2 + b^2.$$

Example 8: Show that $A = (1,0,2)$, $B = (-2,3,1)$ and $C = (0,4,-2)$ are the vertices of a right triangle.

Solution: Find the lengths of the three sides of the triangle:

$$D_{A,B} = \sqrt{(1-(-2))^2 + (0-3)^2 + (2-1)^2} = \sqrt{3^2 + (-3)^2 + 1^2} = \sqrt{19},$$

$$D_{A,C} = \sqrt{(1-0)^2 + (0-4)^2 + (2-(-2))^2} = \sqrt{1^2 + (-4)^2 + 4^2} = \sqrt{33},$$

$$D_{B,C} = \sqrt{(-2-0)^2 + (3-4)^2 + (1-(-2))^2} = \sqrt{(-2)^2 + (-1)^2 + 3^2} = \sqrt{14}.$$

The length of the segment $\overline{AC}$ is the longest, and we use the Pythagorean Formula:

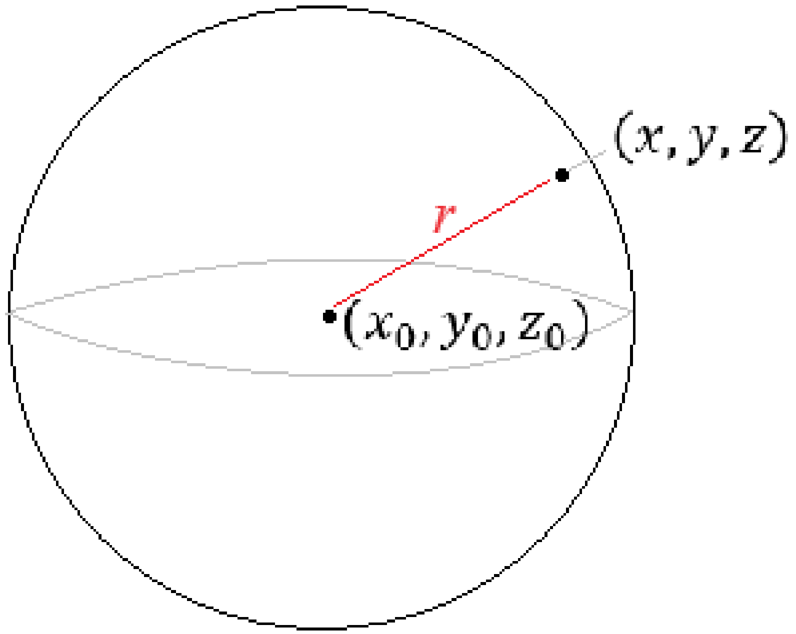
$$\left(\sqrt{33}\right)^2 = \left(\sqrt{19}\right)^2 + \left(\sqrt{14}\right)^2.$$

Since 33 = 19 + 14 is true, the triangle formed by *A*, *B* and *C* is a right triangle.

Spheres and Ellipsoids

A **sphere** is a set of ordered triples (x, y, z) that are of a fixed distance from a single fixed point (x_0, y_0, z_0), called the **center**, and the distance is called the **radius**, r. Using the distance formula, the simplified formula for a sphere can be written as

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$



Example 9: Find the equation of a sphere on which the two points $A = (4,1,-1)$ and $B = (6,7,9)$ lie directly opposite one another (that is, the line through them forms a **diameter** of the sphere. Such points are called **antipodal** points).

Solution: The center is the midpoint of *A* and *B*:

$$M_{A,B} = \left(\frac{4+6}{2}, \frac{1+7}{2}, \frac{-1+9}{2}\right) = (5,4,4).$$

The distance from the midpoint to point *A* is:

$$D_{M,A} = \sqrt{(5-4)^2 + (4-1)^2 + \left(4-(-1)\right)^2} = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}.$$

This is the radius, and since $r = \sqrt{35}$, then $r^2 = 35$. Thus, the sphere is

$$(x-5)^2 + (y-4)^2 + (z-4)^2 = 35.$$

Example 10: Find the center and radius of the sphere $x^2 + 2x + y^2 - 6y + z^2 + 4z = 22$.

Solution: Complete the square three times:

$$x^2 + 2x + \qquad + y^2 - 6y + \qquad + z^2 + 4z + \qquad = 22$$

$$\underbrace{x^2 + 2x + \mathbf{1}}_{(x+1)^2} + \underbrace{y^2 - 6y + \mathbf{9}}_{(y-3)^2} + \underbrace{z^2 + 4z + \mathbf{4}}_{(z+2)^2} = \underbrace{22 + \mathbf{1} + \mathbf{9} + \mathbf{4}}_{36}.$$

Simplified, we have

$$(x + 1)^2 + (y - 3)^2 + (z + 2)^2 = 36.$$

Thus, the sphere has a center of $(-1,3,-2)$ and a radius of $r = \sqrt{36} = 6$.